Algebra

07/11/2019.

Recall:
- Busy with Going Down theorem (harder than Going Up).
- Defd integral closure of an ideal.
- Sketched proof of Going Down.

Recall we're proving the Going Down theorem.

$P_2 \subsetneq P_1$ $\qquad\qquad$ Q_1

$\in \operatorname{Spec} R$ $\qquad\qquad$ $\in \operatorname{Spec} T$.

We defined $S = S_1 S_2$, $\quad S_1 = T \setminus Q_1$, $\quad S_2 = R \setminus P_2$.

Localised.

Assume $TP_2 \cap S = \phi$

In which case $S^{-1}(TP_2)$ is a proper ideal of $S^{-1}T$.

Took maximal ideal containing $S^{-1}(TP_2)$.

It's of the form $S^{-1}Q_2$ for some prime Q_2 of T with $Q_2 \cap S = \phi$.

$Q_2 \cap S = \phi \implies Q_2 \subseteq Q_1$ since $S \supseteq S_1$

and also $Q_2 \cap S = \phi \implies Q_2 \cap R = P_2$

(Note $P_2 \subseteq TP_2 \cap R \subseteq Q_2 \cap R$) $\qquad$ Note ① $TP_2 \subseteq Q_2$

$S^{-1}(TP_2) \subseteq S^{-1}Q_2$.

and since $Q_2 \cap S = \phi$ and $\underset{R \setminus P_2}{S_2} \subseteq S$, we have $P_2 = Q_2 \cap R$.

Now prove $TP_2 \cap S = \phi$:

Take $x \in TP_2 \cap S$. By (3.22), x is in the integral closure of P_2 in T. So by (3.23) it is algebraic over the field of fractions ~~of~~ K of R and its minimal polynomial

$\underline{X^n + r_{n-1}X^{n-1} + \dots + r_0}$

has coefficients in $\sqrt{P_2} = P_2$ (since P_2 prime).

But $x \in S$, and so of the form rt with $r \in S_2$, $t \in S_1$. So $t = \frac{x}{r}$ has minimal polynomial
$$X^n + \frac{r_{n-1}}{r} X^{n-1} + \dots + \frac{r_0}{r^n}.$$

and these coefficients are in R ((3.23) with $I = R$) since t is integral over R.
Write these coefficients as r_i'.
But $r_i \in P_2$ and $r \notin P_2$. and $r_i' = \frac{r_i}{r^{n-i}} \in R$.
$$r_i = r_i' r^{n-i} \in P_2 \implies r_i' \in P_2$$
using primeness of P_2.

So each $r_i' \in P_2$. Hence by definition, t is integral over P_2, and so (3.22), t is in $\sqrt{(TP_2)}$.
This is a <u>contradiction</u>, since $t \in S_1 = T \setminus Q_1$

$TP_2 \subseteq Q_1$ and Q_1 is prime and so $\sqrt{(TP_2)} \subseteq Q_1$
(in Q_1 and not in Q_1).

So x cannot exist.

Finally managed to prove Going Down! ☺

The key result allowing us to use integral extensions is:
(3.24) (Noether's normalisation lemma)
- Let T be a f.g. k-algebra.
- Then T is integral over a subalgebra $k[x_1, \dots, x_r]$ with $x_1, \dots, x_r$ <u>algebraically independent</u> over k.

(3.25) (Def^n) $x_1, \dots, x_r$ algebraically independent over k if the map $k[X_1, \dots, X_r] \longrightarrow k[x_1, \dots, x_r]$
$$X_i \longmapsto x_i$$

is a ring isomorphism.
(no relations between any elements).

$T = k[a_1, \dots, a_n]$

Proof is by induction on n, the no. of generators.
If all the a_i's are algebraic over k, then we can take our subalgebra to be k.

Otherwise, renumber so that $a_1, \dots, a_r$ are algebraically independent over k, and $a_{r+1}, \dots, a_n$ are algebraically dependent over $a_1, \dots, a_r$.

Take $\underset{\neq 0}{f} \in k[X_1, \dots, X_r, X_n]$ such that $f(a_1, \dots, a_r, a_n) = 0$

Thus $f(X_1, \dots, X_r, X_n)$ is a sum of terms

$\lambda_{\underset{\sim}{l}} X_1^{l_1} \dots X_r^{l_r} X_n^{l_n}$ $\quad \underset{\sim}{l} = (l_1, \dots, l_r, l_n)$
$\lambda_{\underset{\sim}{l}} \in k$

Claim: $\exists$ positive integers $m_1, \dots, m_r$ such that:
$\phi : \underset{\sim}{l} \longmapsto m_1 l_1 + \dots + m_r l_r + l_n$
is injective for those $\underset{\sim}{l}$ for which $\lambda_{\underset{\sim}{l}} \neq 0$.

Proof: There are finitely many possibilities for differences $\underset{\sim}{d} = \underset{\sim}{l} - \underset{\sim}{l}'$ with $\lambda_{\underset{\sim}{l}} \neq 0 \neq \lambda_{\underset{\sim}{l}'}$

Write $\underset{\sim}{d} = (d_1, \dots, d_r, d_n)$ and consider the finitely many $(d_1, \dots, d_r) \in \mathbb{Z}^r$ obtained. Vectors orthogonal to these finitely many r-tuples lie in finitely many $(r-1)$ dimensional vector subspaces of $\mathbb{Q}^r$

Pick $(q_1, \dots, q_r)$ with each $q_i > 0$ so that $\sum q_i d_i \neq 0$ for all the finitely many non-zero $(d_1, \dots, d_r)$

Multiply by a positive integer to get $(m_1, \dots, m_r)$ so that $|\sum m_i l_i| > |d_n|$ for all the finitely many $(d_1, \dots, d_r, d_n) = \underline{d}$ with $(d_1, \dots, d_r) \neq \underline{0}$.

Then, if $\phi(\underline{l}) = \phi(\underline{l}')$ then $d_1 = \dots = d_r = 0$ and so $l_n = l_n'$ and $\underline{l} = \underline{l}'$. $\square$

- Back to the proof:

Now put $g(X_1, \dots, X_r, X_n) = f(X_1 + X_n^{m_1}, \dots, X_r + X_n^{m_r}, X_n)$

(the first r things, we're adding on powers of X_n).

This is a sum:

$$\sum_{\underline{l} \text{ s.t. } \lambda_{\underline{l}} \neq 0} \lambda_{\underline{l}} (X_1 + X_n^{m_1})^{l_1} \cdots (X_r + X_n^{m_r})^{l_r} X_n^{l_n}$$

Different terms have different powers of X_n and there will be a single term of highest power in X_n. As a polynomial in X_n, the leading coefficient is one of the $\lambda_{\underline{d}} \in k$.

Put $b_i = a_i - a_n^{m_i}$ for $1 \leq i \leq r$.
and $h(X_n) = g(b_1, \dots, b_r, X_n)$
This has leading coefficient in k and the coefficients are in $k[b_1, \dots, b_r]$.

Moreover $h(a_n) = g(b_1, \dots, b_r, a_n) = f(a_1, \dots, a_r, a_n) = 0$
Dividing through by the top coefficient shows that a_n is integral over $k[b_1, \dots, b_r]$

So for each i, $1 \le i \le r$, $a_i = b_i + a_n^{m_i}$ is also <u>integral</u> over $k[b_1, \ldots, b_r]$.
(integrally closed under ring operations).

Hence, T is integral over $k[b_1, \ldots, b_r, a_{r+1}, \ldots, a_{n-1}]$

<u>fewer</u> generators

By induction hypothesis, $k[b_1, \ldots, b_r, a_{r+1}, \ldots, a_{n-1}]$ is integral over some polynomial subalgebra. (By transitivity of integrality), So T is integral over <u>polynomial</u> subalgebra.

<u>Remark</u>: There are other proofs, not using the little result from linear algebra, using $X_i + X_n^{m_i}$ with m_i being powers of some integer d.
(Nagata).

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