Algebra

09/11/2019.

Prev: Noether's normalisation. If got affine variety, it covers affine space. Will do transcendence degree, show dimensions are equivalent.

As in linear algebra, in a field with subfield k, we can consider maximal algebraically independent subsets over k. They exist by Zorn's lemma. (union of chain works)

They all have the same cardinality — we can use a version of the exchange Lemma. A max. algebraically independence over k is a transcendence basis of K over k. tr. $\deg_k(K)$ ← Notation.

The algebraic closure of a subset S over k is the set of elements which are algebraically dependent over the subfield generated by k and S.
(look at Stewart's Galois Theory 151 - 153)
(although most books will have this)

(3.26) Theorem

Let T be a f.g. k-algebra that is an integral domain. Let L be its fraction field. Then $\dim T = \text{tr.} \deg_k(L)$.

In particular $\dim k[X_1, \dots, X_n] = \text{tr.} \deg_k k(X_1, \dots, X_n) = n$

Proof: Apply Noether normalisation (3.24). T is integral over $k[x_1, \dots, x_r]$ with $x_1, \dots, x_r$ algebraically independent. By (3.19) Corollary of Going Up, $\dim T = \dim k[x_1, \dots, x_r]$

Thus any f.g. k-algebra has dimension = dimension of a polynomial algebra with r variables with $r = \text{tr. deg}_k L$

It's left to prove that dim (poly. algebra in r variables) $= r$.
(Recall, we observed $\dim k[x_1, \dots, x_r] \geq r$ polynomial algebra by writing down a chain of primes of length r).

We prove this by induction:
$r = 0$ ✓ (easy).
Now, if $P_0 \subsetneq \dots \subsetneq P_s$ is a chain of prime ideals, we may assume $P_0 = \{0\}$, $P_1 = (f)$ principal ideal. (f irreducible)

(since we know that polynomial algebras are UFDs).
Recall, UFDs are where any non-zero prime contains a principal one.

But $\text{tr. deg}\left(\text{ff}\left(k[x_1, \dots, x_r]/(f)\right)\right) = r - 1$
(f.f means fraction field).
However $\dim\left(k[x_1, \dots, x_r]/(f)\right) = \dim k[\overline{Y}_1, \dots, \overline{Y}_t]$
with $\overline{Y}_1, \dots, \overline{Y}_t$ algebraically independent in $k[x_1, \dots, x_r]/(f)$ with $k[x_1, \dots, x_r]/(f)$

integral over $k[\overline{Y}_1, \dots, \overline{Y}_t]$ (by Noether normalisation)

However, $\text{tr. deg}_k\, k(\overline{Y}_1, \dots, \overline{Y}_t) = \text{tr. deg}\left(\text{f.f } k[x_1, \dots, x_r]/(f)\right)$
So $t = r - 1$.
By induction: $\dim k[\overline{Y}_1, \dots, \overline{Y}_t] = r - 1$

But $P_1 = (f)$. $\overline{P_1} \subsetneq \overline{P_2} \subsetneq \dots \subsetneq \overline{P_s}$ is a chain of primes of length $s - 1$ (writing 'bars' for mod P_1)

So $s-1 \le r-1$ and $s \le r$, Hence,
$\dim k[x_1, \dots, x_r] \le r$, and so we have equality.

Last result of chapter:

(3.27) Theorem:

Let R be a Noetherian ring, and integral domain, integrally closed, K its field of fractions. L finite separable field extension of K.

Let T_1 be integral closure of R in L. Then T_1 is a f.g. R-module.
(Note: separability always applies in characteristic 0)

(3.28) Corollary: Set $R = \mathbb{Z}$. Then the integral closure of $\mathbb{Z}$ in a number field L (i.e. a finite field extension of $\mathbb{Q}$) is a f.g. $\mathbb{Z}$-module.
(so is abelian group)

(3.29) Corollary: Let $\operatorname{char} k = 0$, and T be a f.g. k-algebra which is an integral domain, integral over $R = k[x_1, \dots, x_r]$ polynomial algebra.

Let $L =$ f.f of T.
Then the integral closure T_1 of R in L is a f.g. R-module.

Thus T_1 is integrally closed.
$$\operatorname{Spec} T_1 \xrightarrow{res} \operatorname{Spec} T \xrightarrow{res} \operatorname{Spec} R$$
and all the fibres are finite.

Proof: Immediate from (3.27) apart from finiteness of the fibres - see Ex. Sheet 2, Q15, Q16.
[Hand-in by 10am Monday!]

$[L:K(x)]$

The proof of (3.27) uses the trace function $\mathrm{Tr}_{L/K}(x) = -[L:K(x)] \cdot$ (next to top coefficient of minimal polynomial of x over K)

for $x \in L$.

for <u>any</u> finite field extension L/K

If L is <u>Galois</u> over K, $\mathrm{Tr}_{L/K}(x) = \sum_{g \in G} g(x)$ where $G = \mathrm{Gal}(L/K)$

(this is the sum of conjugates of x, but we may have repetitions, and hence the factor $[L:K(x)]$ – the coefficient from the minimal polynomial is the sum of the conjugates)

<u>Quote:</u> If L is separable over K, then

$$L \times L \longrightarrow K$$
$$(x,y) \longmapsto \mathrm{Tr}_{L/K}(xy)$$

is a <u>non-degenerate</u> symmetric K-bilinear form on L.
(e.g. Reid 8.13)
(might not be in Atiyah-MacDonald)

Proof of (3.27)
Pick a K-vector space basis of L, say $y_1, \dots, y_n$
By multiplying by suitable elements of K, we may assume each y_i lies in T_1.
Since $\mathrm{Tr}_{L/K}(xy)$ yields a non-degenerate symmetric bilinear form, we can find a basis $x_1, \dots, x_n$
$\mathrm{Tr}(x_i y_j) = \delta_{ij}$.

We'll show that $T_1 \subseteq \sum R x_i$ (since R Noetherian, we deduce T_1 is a f.g. R-module)

Let $z \in T_1$. Then $z = \sum \lambda_i x_i$ with $\lambda_i \in K$

So $Tr(zy_j) = Tr(\sum \lambda_i x_i y_j)$

$= \sum \lambda_i \, Tr(x_i y_j) = \sum \lambda_i \delta_{ij} = \lambda_i$

But z and y_j are in T_1, and hence ~~$z y$~~ $zy_j \in T_1$

By (3.23) with $I = R$ (using R integrally closed), the coefficients of min. polynomial of zy_j lie in R. So next to top coefficient is in R.

So $Tr(zy_j) \in R$. So $\lambda_j \in R$.

[Will talk about heights next time].