Algebra 12/11/2019

Reminder: Ex. Class this afternoon, 2pm.

NB: E-mail me to prompt discussion of particular questions brookes@dpmms.cam.ac.uk.

§4: Heights.

R commutative with a 1.
but note we're going talking about general Noetherian R (not necessarily integral domain), not just f.g. k-algebras.

Recall that for UFDs the height one primes are precisely those of the form (f) with f irreducible.
The Principal Ideal theorem gives a recipe for finding height 1 primes in general.

(4.1) Principal Ideal Theorem (Krull)
Let R be a Noetherian ring (commutative) and a be a non-unit. Let P be a minimal prime over (a). Then $\mathrm{ht}\,P \leq 1$.

This starts an inductive argument, and appears in the proof of the inductive step proving:

(4.2) (Generalised principal ideal theorem)
Let R be a Noetherian ring, I proper ideal, generated by n elements. Then $\mathrm{ht}\,P \leq n$ for every minimal prime over I.

(4.1 is simply $n=1$)

This gives

(4.3) Corollary:

a) Every prime ideal P of a Noetherian ring R has finite height, $\mathrm{ht}\, P \le$ min. no of generators of P.

b) Every Noetherian local ring has finite dimension. (for local rg, dim = ht of max ideal)
$\le$ min. no of generators of the unique maximal ideal P
$= \dim_{R/P} (P/P^2)$ R/P - vector space dimension of P/P^2.

Proof of (4.3):

a) Any ideal of a Noetherian ring is f.g, and a prime ideal is minimal over itself.
So $\mathrm{ht}(P) \le$ min. no of generators of P.

b) for a local ring R, $\dim R = \mathrm{ht}\, P$
P unique max ideal.
By a) $\dim R = \mathrm{ht}\, P \le$ min. no of generators of P.
The final equality follows from Nakayama's lemma.

Claim: P is generated by $x_1, \dots, x_r \iff P/P^2$ is generated by $\bar{x}_1, \dots, \bar{x}_r$ writing $\bar{\ }$'s for images in P/P^2.

Proof: $(\Rightarrow)$ ✓

$(\Leftarrow)$ Suppose $\bar{x}_1, \dots, \bar{x}_r$ generate P/P^2 with $x_i \in P$ (pullback)
Consider $I = (x_1, \dots, x_r) \subseteq P$.
Clearly $I + P^2 = P$ and so $P(P/I) = P/I$
(Note: $P = \mathrm{Jac}(R)$, R local)

$\therefore$ By Nakayama $\Rightarrow$ $P/I = 0$
$\therefore$ P is generated by $x_1, \dots, x_r$

(4.4) Defn: A regular local ring R is one where
$\dim R = \dim_{R/P} (P/P^2)$.

In geometry, this corresponds to localisation at a non-singular point.
$P/P^2 \equiv$ cotangent space at point.

Remark: (4.8) $\dim R = \mathrm{ht}\, P \leq$ min. no of generators of I with $\sqrt{I} = P$.
a local ring R.

In fact, but not proved here:
$\dim R = \mathrm{ht}\, P =$ min. no of generators of some I with $\sqrt{I} = P$.
(strong link between no. of gens and height).

Proof of (4.1): Principal Ideal Theorem.
Take non-unit $a \in R$ and P is minimal over (a).
First localise at P, R_P has unique max ideal
$P_P = S^{-1}P$ where $S = R \setminus P$. Observe that
$P_P = S^{-1}P$ is minimal over $S^{-1}(a)$, $\mathrm{ht}(S^{-1}P)$
$= \mathrm{ht}(P)$.
(follows from correspondence of primes under localisation).

So we may assume that R is local with unique maximal ideal P (we'll need another localisation and so we'll have a different mult. closed subset S. later)

Suppose $\mathrm{ht}\, P > 1$ and we have a chain of primes
$Q' \subsetneq Q \subsetneq P$, and aim for a contradiction.

Consider $R/(a)$. This has a unique maximal ideal $P/(a)$ which is also a <u>minimal</u> prime. So it is the only prime of $R/(a)$

Thus $\mathcal{N}(R/(a)) = P/(a)$ is nilpotent (as $R/(a)$ is <u>Noetherian</u>). So $P^n \subseteq (a)$ for some n.

Now consider $R \supsetneq P \supsetneq P^2 \supseteq \dots \supseteq P^n$
and each factor is a <u>finite dimensional</u> R/P-vector space

Each factor satisfies <u>descending chain condition</u> [DCC] on subspaces and so R/P^n satisfies DCC on submodules/ideals. R/P^n is <u>Artinian</u> and so $R/(a)$ is <u>Artinian</u>. (quotient of Artinian is Artinian).

Now consider localisation at Q, with $S = R \setminus Q$.
$S^{-1}R \supsetneq S^{-1}Q \supseteq (S^{-1}Q)^2 \supseteq (S^{-1}Q)^3 \supseteq \dots\dots$
is a chain of ideals in $S^{-1}R$.

By (1.16) (i) we can set $I_m = \{ r : \frac{r}{1} \in (S^{-1}Q)^m \}$
and $S^{-1} I_m = (S^{-1}Q)^m$.

Clearly, $\underset{\uparrow (1.16)}{Q = I_1} \supseteq I_2 \supseteq \dots$ (*)

and hence $I_1 + (a)/(a) \supseteq I_2 + (a)/(a) \supseteq \dots$
is a descending chain of ideals in $R/(a)$.

But $R/(a)$ satisfies DCC, and so this chain terminates
$I_m + (a) = I_{m+1} + (a)$ for some m.

Now, we show that (*) terminates

$\longrightarrow$

Take $r \in I_m$, Then $r = t + xa$ for some $t \in I_{m+1}$, $x \in R$.

So $xa = r - t \in I_m$
However $a \notin Q$ (as P minimal prime over (a))
and so $a \in S$ and $a/1$ is a unit in $S^{-1}R$.

$$xa \in I_m \iff xa/1 \in (S^{-1}Q)^m$$
$$\iff x/1 \in (S^{-1}Q)^m$$
$$\iff x \in I_m$$

So $I_m = I_{m+1} + I_m a$

and: $I_m/I_{m+1} = P(I_m/I_{m+1})$ since $a \in P$.

Nakayama $\Rightarrow$ $I_m/I_{m+1} = 0$ $\qquad P = \mathrm{Jac}\, R$.
Hence $I_m = I_{m+1}$, and $(*)$ terminates.

Now $(S^{-1}Q)^m = S^{-1}I_m$ from (1.16) (c).
$(S^{-1}Q)^{m+1} = S^{-1}I_{m+1}$

So $(S^{-1}Q)^m = (S^{-1}Q)^{m+1}$

Can use Nakayama again! $S^{-1}Q = \mathrm{Jac}(S^{-1}R)$
and so Nakayama $\Rightarrow$ $(S^{-1}Q)^m = 0$.

Thus $S^{-1}Q$ nilpotent. However when we localise the chain $Q' \subsetneq Q$, we get $S^{-1}Q' \subsetneq S^{-1}Q$ primes in $S^{-1}R$, gives contradiction as $(S^{-1}Q)^m = 0$. !!

This proves the theorem. $\square$

(Will take about another 15 min to prove general theorem).