Algebra

14/11/2019

Proof of (4.2) Generalised Principal Ideal Theorem

(Last time finished Krull's PIT, use it for Q>0 in Ex Sheet 2. Most of the work done for Krull's PIT)

R Noetherian, I proper ideal gen. by n elements, P minimal prime over I. Then $ht(P) \leq n$.

Proof by induction on n.

$n = 1$ ✓ Done by 4.1.

Assume $n > 1$, we may assume by passing to R_P, that R is local, with unique maximal ideal P.

Pick any prime Q maximal subject to $Q \subsetneq P$. and thus P is the only prime strictly containing Q.

Claim: $ht\, Q \leq n-1$

(It's enough to do this for all such Q in order to prove $ht\, P \leq n$).

Since P is minimal over I, $Q \not\supseteq I$.

By assumption $I = (a_1, \ldots, a_n)$ say and we may assume $a_n \notin Q$.

P is the only prime containing $Q + (a_n)$ and so $N(R/Q+(a_n)) = P/Q+(a_n)$. is nilpotent

(R Noetherian) and so there is m such that $a_i^m \in Q + (a_n)$ for all $i \leq n-1$.

$= t_i + x_i a_n$ for some $t_i \in Q$
$x_i \in R$.

Any prime of R containing $t_1, \dots, t_{n-1}$ and a_n contains $a_1, \dots, a_n$.

Note: $(t_1, \dots, t_{n-1}) \subseteq Q$ since $t_i \in Q$.

<u>Claim</u>: Q is minimal over $(t_1, \dots, t_{n-1})$.
write $\bar{R} = R/(t_1, \dots, t_{n-1})$

bars for images in $\bar{R}$.

The unique maximal ideal of $\bar{R}$ is $\bar{P}$, is a minimal prime over $(\bar{a_n})$.

Apply (4.1) to get $ht(\bar{P}) \leq 1$.
But $\bar{Q} \subsetneq \bar{P}$ and so $ht(\bar{Q}) = 0$.
and so Q is minimal over $(t_1, \dots, t_{n-1})$.

So we may apply inductive hypothesis to give $ht Q \leq n-1$.

(Slightly used PIT twice, but much extra effort)

End of chapter.

Should now cover Artinian rings (in <u>non-commutative</u> case).
2 lectures of non-commutative, then back to commutative

§5: Artinian rings

In this chapter, R need not be commutative (but still have unity).

(5.1) **Defn**: R is right Artinian if it satisfies the descending chain on right ideals (similarly for modules, left Artinian).

Examples (1) k field, any R that is finite dimensional as a k-vector space is right and left Artinian.

- right ideals, left ideals are vector subspaces.

e.g. $M_n(k)$ $n \times n$ matrices over k.

Division rings ~~that finite dimensional over k~~. actually don't need this.

e.g. $\mathbb{H}$ quaternions $\mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$

$$\begin{cases} i^2 = j^2 = k^2 = -1 \\ ij = k, \quad ji = -k \end{cases}$$

division ring

$M_n(D)$ for division ring D.

right ideal generated by matrix A.

$AR = \{B : \text{columns of } B \leq \text{right span of columns of } A\}$

scalar multiplication on right.

Right ideals are of form
$I_V = \{B : \text{columns of } B \leq \text{right subspace } V \text{ of } D^n \}$

column vectors

Similarly for left ideals
$RA = \{B : \text{rows of } B \leq \text{left span of rows of } A\}$.

Left ideals are of form
$J_V = \{B : \text{rows of } B \leq \text{left subspace } V \text{ of } D^n\}$

row vectors.

Note that only 2-sided ideals are 0 and the whole of $M_n(D)$

(5.2) (Def^n) R is a <u>simple ring</u> if the only 2-sided ideals are 0 and R.

Thus, $M_n(D)$ is a simple ring. ☺.

Example: $R = \left\{ \begin{pmatrix} q & r \\ 0 & s \end{pmatrix} ; q \in \mathbb{Q}, r, s \in \mathbb{R} \right\}$

is <u>not</u> left Artinian but is right Artinian. (Can come up with examples where left, not right Art, but doesn't come up very often).

kG, k field G finite group basis labelled by $g \in G$:

$\left(\sum_{g \in G} \lambda_g g\right)\left(\sum_{h \in G} \mu_h h\right) = \left(\sum_{k \in G} \nu_k k\right)$ where

$$v_k = \sum_{\substack{g,h \\ \text{s.t. } gh=k}} \lambda_g \mu_h$$

(5.3) <u>Def^n</u>: The Jacobson radical <u>Jac R</u> is the intersection of the maximal <u>right</u> ideals.
(will show get same thing for left)

<u>Remark</u>: This is actually a 2-sided ideal.

I is a maximal right ideal $\Longleftrightarrow$ R/I is a simple right R-module.

(5.4) <u>Def^n</u>: A simple module M is one where the only submodules are 0 and M.
Let M be a simple right R-module and $m \neq 0$, $m \in M$. Then $\mathrm{Ann}_R(m) = \{r : mr = 0\}$ is a maximal right ideal of R.
(Note annihilators of elements are right ideals)

However $\mathrm{Ann}_R M = \bigcap_{m \in M} \mathrm{Ann}(m)$ is a 2-sided ideal,
since if $r \in \mathrm{Ann} M$, $x \in R$,

$m(xr) = (mx)r = 0$ and so $xr \in \mathrm{Ann}(m)$ for all m.

We can see that $\mathrm{Jac} R = \bigcap_{\substack{M \text{ simple} \\ \text{right } M\text{-module}}} \mathrm{Ann}(M)$ and so a 2-sided ideal.

(3.5) Nakayama's Lemma. — very useful!!
the following are equivalent for a right ideal I.

(1) $I \leq \operatorname{Jac} R$
(2) If M f.g. R-module with submodule N satisfying $N + MI = M$ then $N = M$.

(3) $\{1 + x : x \in I\} = G$ is a subgroup of unit group of R.

Proof: Ex Sheet. (which is why original proof we did was not A,M proof)

From this, we can see that $\operatorname{Jac} R$ is characterised as the largest 2-sided ideal such that $\{1 + x : x \in \operatorname{Jac} R\}$ form a subgroup of the unit group.

This characterisation does not include right/left.

If we developed the definition of $\operatorname{Jac} R$ using left ideals, we'd get the same ideal.

(3.6) Def^n: R is semisimple if $\operatorname{Jac} R = \{0\}$

Examples: ① $M_n(D)$ for D division ring. is semisimple

② G = cyclic order p, $\mathbb{F}_p$ field of p elements

$\mathbb{F}_p(G) \cong \mathbb{F}_p[X]/(X^p - 1)$ not semisimple since $X^p - 1 = (X-1)^p$.

③ kG, G finite, char $k=0$ is semisimple (needs proof, not trivial).

Our main goal is to prove Artin-Wedderburn theorem which says a right Artinian semisimple ring is a direct sum of matrix algebras over division rings.

⟶ . (will take abt 30 min to prove).

[will do some homological algebra afterwards]