Algebra 16/11/2019.

Recall: Thinking about Artinian rings in the more general context. Did matrix rings over division rings.
Vector spaces over division rings → same theory holds, just remember which side scalar multiplication is done. Defined $\mathrm{Jac}(R)$ in non-com case. Going to Artin-Wedderburg.

(5.7) Theorem Let R be right Artinian. Then (i) $\mathrm{Jac} R$ is nilpotent,
(ii) R is right Noetherian,

To prove this, we need some lemmas and terminology.

(5.8) Lemma: Let R be a semisimple, right Artinian. Then R is a finite direct sum of simple right R-modules.

Proof: $(0) = \mathrm{Jac} R = \bigcap$ (maximal right ideals).

Consider maximal right ideals M_i and the chain: $R \gneq M_1 \geq M_1 \cap M_2 \geq \ldots$

The DCC on right ideals forces this to terminate. $(0) = \mathrm{Jac} R = M_1 \cap \ldots \cap M_n$ say, and we may assume n is minimal.

Consider the R-module map:

$$R \longrightarrow \bigoplus_{i=1}^{n} \left(R/M_i\right)$$

$$r \longmapsto (r+M_1, r+M_2, \ldots, r+M_n)$$

Note R/M_i are <u>simple modules</u>.

Consider the restriction of $R \longrightarrow R/M_i$ to $\bigcap_{j \neq i} M_j$ non-zero by our assumption.

This is injective, since kernel is $\bigcap M_i = (0)$. surjective since image is a non-zero submodule of simple module M/M_i.

Image of $\bigcap_{j \neq i} M_j$ under θ is $(0, \ldots, 0, R/M_i, 0, \ldots, 0)$

Hence image of θ is $\bigoplus_{i=1}^{n} R/M_i$

Note θ is injective since $\ker \theta = \bigcap M_i = (0)$.

(5.9) <u>Lemma</u>: Let R be a semisimple right Artinian, and M is a right Artinian R-module. Then, M is a finite direct sum of simple R-modules

<u>Proof</u>: Ex. Sheet 3

(These ex sheets will be slightly reordered from last year)

(5.10) <u>Def^n</u>: The <u>socle</u> of M (M non-zero Artinian module) is the sum of all the simple submodules of M. Note that since $M \neq 0$, Artinian, it does have minimal non-zero submodules, which are necessarily simple. Thus $\text{soc}\, M \neq 0$.

(Alt defn of soc: Those in M killed by $\text{Jac} R$)

(5.11) <u>Lemma</u>: $\text{Soc}\, M = \{ m \in M : mJ = 0 \}$ where $J = \text{Jac} R$.

<u>Proof</u>: Each minimal submodule of M is simple and therefore of the form $R/\text{Ann}(m)$ for each $m \in M'$.

So $J \leq \bigcap_{m \in M'} \text{Ann}(m)$ and so $\text{Jac} R$ annihilates M' and hence annihilates $\text{Soc}\, M$. Conversely, if $mJ = 0$, then mR can be regarded as a R/J-module. But mR inherits Artinian property and so we have Artinian module for semisimple ring R/J.

So (5.9) $\Rightarrow$ mR is a finite direct sum of simple modules. So $mR \subseteq \text{soc}\, M$.

(5.12) <u>Defn</u>: We define the <u>socle series</u> of M inductively. $\text{soc}_0(M) = (0)$, $\text{soc}_1(M) = \text{soc}(M)$.

$$\text{soc}_i(M) / \text{soc}_{i-1}(M) = \text{soc}\left(M / \text{soc}_{i-1} M \right)$$

We get an <u>ascending</u> series:

$$(0) \quad \text{soc}_0 M < \text{soc}_1 M < \text{soc}_2 M < \dots$$

Note that we have strict inequalities until we reach $\text{soc}_n(M) = M$.

We do reach M since we have <u>descending</u> chain.

$$R \underset{\neq}{\geq} J \underset{\neq}{\geq} J^2 \geq \dots$$

which must terminate $J^n = J^{n+1}$, and so

$$\begin{aligned} soc_n(M) &= \{m \in M : mJ^n = 0\} \\ &\parallel \\ soc_{n+1}(M) &= \{m \in M : mJ^{n+1} = 0\}. \end{aligned}$$

(note: $soc_i(M) = \{m \in M : mJ^i = (0)\}$)

<u>Proof of (5.7):</u>

Consider $R > J \geq J^2 > \dots$.

This must terminate and so we do have the socle series reaching M. Each factor is annihilated by J and so can be viewed as an R/J-module.

$M = R$ $\quad soc_i(M)/soc_{i-1}(M)$ is a finite direct sum of simple modules.

Such a finite direct sum satisfies both ACC and DCC on submodules, and so $soc_i(M)/soc_{i-1}(M)$ is a <u>right Noetherian</u> module.

So $soc_n(M)$ is right Noetherian
$\parallel$
M

Note $soc_n(R) = R \Rightarrow R$ is annihilated by J^n
$\Rightarrow 1$ is annihilated by J^n
$\Rightarrow J^n = (0)$.

$\square$

To prove Artin-Wedderburn, we need to think about endomorphism rings of modules.

(5.13) Schur's Lemma (nothing to do with mult. closed)

Let S be a simple right R-module.
Then $End_R(S)$ is a division ring.

If S_1 and S_2 are non-isomorphic simple modules then $Hom_R(S_1, S_2) = \{0\}$.
Note that S is a left $End_R(S)$-module.
and thus S is a $End_R(S) - R$ bimodule

Proof: Let $\phi : S \longrightarrow S$ R-module map.
Then either $\phi(S) = 0$ and $\phi = 0$.
or $\phi(S) = S$. since S simple.

Furthermore, $\ker\phi$ is a submodule of S.
Either $\ker\phi = 0$ or $\ker\phi = S$ in which case $\phi = 0$.

So, if $\phi \neq 0$ it must be bijective and have a right and left inverse.
Thus $End_R(S)$ is a division ring (don't necessarily have commutativity)

If $S_1 \not\cong S_2$ simple and $\phi : S_1 \longrightarrow S_2$ a similar argument about $\ker\phi$ and $\operatorname{im}\phi$ shows $\phi = 0$.

(5.14) Lemma: Regard R as a right R-module R_R
Then $End_R(R_R) \cong R$.

Proof: Observe that multiplication on left by $r \in R$ gives a R-module endomorphism of R_R

Observe that $\phi \in End_R(R_R)$ is uniquely determined by $\phi(1)$ and so:

$$End_R(R) \longrightarrow R$$
$$\phi \longmapsto \phi(1)$$

is the sought after isomorphism.

(5.15) Artin - Wedderburn Theorem NB!

Let R be a semisimple, right Artinian ring.

Then $R = \bigoplus_{i=1}^{r} R_i$ where $R_i = M_{n_i}(D_i)$ for division ring D_i. matrix rings

and the R_i are uniquely determined.

R has exactly r isomorphism classes of simple modules S_i, and $End_R(S_i) = D_i$ and $\dim_{D_i} S_i = n_i$ (can view S_i as a left D_i-vector space). Furthermore if R is finite-dimensional as a k-vector space for a field k, then D_i have finite dimension as k-vector spaces.

If k is algebraically closed then $D_i \cong k$, and so for example $\mathbb{C}G$ for a finite group $= \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})$ where $r =$ no. of simple modules up to $\cong$.

$\shortparallel$

dimension $= M_i$