# Algebra

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Recall: (5.15) Artin-Wedderburn

- $R$ semisimple right Artinian $\qquad R = \bigoplus_{i=1}^{r} R_i$ unique
- $R \cong M_{n_i}(D_i)$ $D_i$ division ring.
- $r$ = no of isomorphism classes of simple modules
  $n_i = \dim_{D_i} S_i$ $\quad$ $S_i$ - simple module corr. to $R_i$.
  $D_i = \mathrm{End}_R(S_i)$.

(5.16) Corollary: $G$ finite group

$Z(\mathbb{C}G)$ is an $r$-dimensional $\mathbb{C}$-vector space
centre $\quad$ where $r$ = no. of IM classes of simple $\mathbb{C}G$-modules.
= no. of conjugacy classes ~~of~~ in $G$.

Proof: Any class sum $\sum_{g' \in ccl(g)} g' \in Z(\mathbb{C}G)$

Any element of $Z(\mathbb{C}G)$ is a linear combination of such class sums. These class sums are also linearly independent over $\mathbb{C}$. So $\dim_{\mathbb{C}} Z(\mathbb{C}G)$ = no. of conjugacy classes.

(haven't yet applied Artin-Wedderburn).

But $\mathbb{C}G$ is semisimple, it Artinian and so Artin-Wedderburn applies. $\mathbb{C}$ is alg. closed and so $\mathbb{C}G = \bigoplus M_{n_i}(\mathbb{C})$.

But $Z(M_n(\mathbb{C})) = \{\lambda I : \lambda \in \mathbb{C}\}$ 1-dimensional.

$Z(\bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}))$ is $r$-dimensional, and from Artin-Wedderburn:

$r$ = no. of simple $\mathbb{C}G$-modules up to $\cong$.

(Note: If char $= p$, then <u>lose semisimplicity</u>)

Proof of Artin-Wedderburn:

$R_R$ is a finite direct sum of simple modules by (5.9). Group those together that are isomorphic to each other.

$$R_R = (S_{11} \oplus \ldots\ldots \oplus S_{1n_1}) \oplus (S_{21} \oplus \text{———} \oplus S_{2n_2}) \oplus \ldots$$

so that $S_{im} \cong S_{il}$ for $1 \leq m, l \leq n_i$
and $S_{im} \not\cong S_{jl}$ if $i \neq j$

(everything in here isomorphic).

Write $R_i = S_{i1} \oplus \text{———} \oplus S_{in_i}$.

Let $S$ be a simple ~~right R-module~~ submodule of $R_R$ and consider the projections: $\pi_{ik} : R \longrightarrow S_{ik}$ restricted to $S$. By Schur's lemma, $\pi_{ik}|_S$ are $0$ or isomorphisms.

But at least one of the restrictions must be non-zero and the non-zero restrictions must all be into the same $R_i$.

Thus $S \leq R_i$ for some $i$ and is <u>isomorphic</u> to all the $R_{ik}$.

If all the $S_{ik} \cong S_i$ say then
$R_i$ = sum of all simple submodules of $R_R$ that are isomorphic to $S_i$, uniquely determined.

Consider $End_R(R_R)$.
By (5.14), we know that $End_R(R_R) \cong R$.
However, $End_R(R_R) = End_R((S_{11} \oplus \dots \oplus S_{1n_1}) \oplus (S_{21} \oplus \dots \oplus S_{2n_2}) \oplus \dots\dots)$

Consider $End_R(S_{11} \oplus \dots \oplus S_{1n_1})$ This is $M_{n_1}(D_1)$ where $D_1 = End_R(S_1)$, this is division ring from Schur's lemma.

$\therefore$ $\phi \in End_R(S_{11} \oplus \dots \oplus S_{1n_1})$ is represented by a matrix $(\phi_{m\ell})$ where $\phi_{m\ell} \in Hom_R(S_{1m}, S_{1\ell})$.

$$\text{So } R = End(R_R) = \left(\begin{array}{c|c|c} M_{n_1}(D_1) & 0 & 0 \\ \hline 0 & M_{n_2}(D_2) & 0 \\ \hline 0 & 0 & \ddots \end{array}\right)$$

Recall our example about $M_n(D)$ for a division ring $D$. We saw that the minimal right ideals consist of matrices $B$ whose columns are all of the form $\begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \lambda$ for a column $\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$, $\lambda \in D$.

These are all of dimension $n$ as a $D$-vector space and so $\dim_{D_i} S_i = n_i$

Example: $kS_3$, $k$ field, $g = (1\ 2)$
$h = (1\ 2\ 3)$

When char $k = 0$, we know that $kG$ is semisimple. There are 3 conjugacy classes.

$U_1$ trivial 1-dimensional - $g$ acts trivially.
$U_2$ 1-dimensional with $g$ acting by $g$ acting by multiplication by $-1$, and $h$ acting by multiplication by $+1$.

$U_3$ 2-dimensional $k^2$ row vectors with $g$, acting by mult. on right by $\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$

and $h$ acts via $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.

These are all simple modules whatever $k$ is with char $k = 0$.

char $k = 3$ $\overline{U_1}$ working (mod 3) 1-dimensional
$\overline{U_2}$ 1-dimensional } simple.

Note that $\overline{U_3}$ in characteristic 3 has $(2\ 1)$ is a common eigenvector of both $g$ and $h$, and so there is a 1-dimensional submodule.

$(2\ 1)\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ $(2\ 1)\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

In fact there are only 2 isomorphism classes of simple modules. $\mathrm{Jac}\, kS_3 = \ker(kS_3 \longrightarrow kC_2)$, map induced by group map $S_3 \longrightarrow C_2$

4-dimensional over $k$.

Semisimple quotient ${}^{kS_3}/_{\mathrm{Jac}(kS_3)} \cong M_1(k) \oplus M_1(k)$

corresponding to the simple 1-dimensional modules

$(h-1)^3 = h^3 - 1 = 0$
since $h$ has order 3.

One can show $\mathrm{Jac}(kS_3)$ is nilpotent.

$\mathrm{Soc}(kS_3)$ = right ideal generated by $(h-1)^2 =$
$= h^2 + h + 1$.

2 dimensional and is a direct sum of a copy of $\overline{U_1}$ and $\overline{U_2}$.

- A taste of modular representation theory! We wouldn't be examined on this, but should do the odd example ☺.