Algebra

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§6: Filtrations and Graded Rings

R commutative with a 1.

(6.1) **Defⁿ**: A ($\mathbb{Z}$)-*filtered ring* R is one with additive subgroups R_i

$$\dots \subseteq R_{-1} \subseteq R_0 \subseteq R_1 \subseteq R_2 \subseteq \dots$$

with $R_i R_j \subseteq R_{i+j}$ for $i,j \in \mathbb{Z}$, $1 \in R_0$.

Note: $\bigcup R_i$ is a subring – usually we have $\bigcup R_i = R$ exhaustive filtration.

R_0 is a subring

$\bigcap_{i \in \mathbb{Z}} R_i$ is an ideal of R_0, usually have $\bigcap R_i = \{0\}$ separated filtration.

Examples: ① I-adic filtration of a ring R where I is an ideal.

$$\begin{cases} R_i = R & \text{for } i \geq 0 \\ R_{-j} = I^j & \text{for } j > 0. \end{cases}$$

If R is a local ring we're particularly interested in the P-adic filtration where P is the maximal ideal.

② If R is a k-algebra generated by $x_1, \dots, x_n$ say set $R_{-j} = 0$ for $j > 0$.

$R_0 = k1$

R_i = k-subspace of polynomial expressions in the generators of total degree $\leq i$

[Q2 of Ex Sheet: Find a k-algebra ....]

(6.2) **Defn**: The associated graded ring

$$gr R = \bigoplus R_i / R_{i-1}$$ as an additive group.

with multiplication:

$$(r + R_{i-1})(s + R_{j-1}) = rs + R_{i+j-1}$$

where $r \in R_i$, $s \in R_j$.

e.g. for P-adic filtration of a local ring

$$gr R = \bigoplus P^j / P^{j+1}$$ ~ j^{th} component.

Write $K = R/P$ then $gr R$ is generated as a K-algebra by any K-vector space basis of P/P^2.

Remark: (1) on the next ex. sheet, we'll see that the regular case is when $gr R$ is a polynomial algebra in n variables where $n = \dim_K (P/P^2)$

(2) filtrations of associated graded rings are also of use for non-commutative R e.g. R = universal enveloping algebra of a finite dimensional Lie algebra. $gr R$ is commutative when R filtered as in Example Sheet 2.

(6.3) **Defn**: Let R be a filtered ring with filtration $\mathcal{R}$ and let M be an R-module.

Then M is a filtered R-module w.r.t. filtration $\mathcal{R}$ of R.

If there are additive subgroups M_i of M, such that $R_j M_i \subset M_{i+j}$.

$$gr\, M = \bigoplus M_i / M_{i-1}$$

$$(r + R_{j-1})(m + M_{i-1}) = rm + M_{i+j-1} \quad \text{for } r \in R_j,\ m \in M_i.$$

Thus $gr\,M$ is a $gr\,R$-module

(6.4) Lemma: If $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ is exact, then $0 \longrightarrow gr\,N \longrightarrow gr\,M \longrightarrow gr(M/N) \longrightarrow 0$ is exact, using the filtration $\{N \cap M_i\}$ for N.

$(M_i + N)/N$ for M/N

Proof: Should be on ex. sheet, before Q10, Q $9\frac{3}{4}$. :)

(6.5) Def^n: The Rees ring of the filtration $\{R_i\}$ of R is a subring of $R[T, T^{-1}]$.

$$Rees(R) = \bigoplus_{j \in \mathbb{Z}} R_j T^j \subseteq R[T, T^{-1}]$$

Note: $1 \in R_0 \subseteq R_1$ and so $T \in Rees(R)$.

Observe that $R = Rees(R)/(T-1)$ ← ideal of $Rees(R)$ generated by $T-1$

$$gr\,R = Rees(R)/(T)$$

We can do corresponding thing for modules!

Given R-modules M with a filtration M_j w.r.t. filtration $\{R_i\}$ of R.

$$\mathrm{Rees}(M) = \bigoplus M_i T^i$$

This is a $\mathrm{Rees}(R)$-module

$$gr(M) = \mathrm{Rees}(M) / T\,\mathrm{Rees}(M)$$

(6.6) <u>Def^n</u>: A ($\mathbb{Z}$)-graded ring $S = \bigoplus_{i \in \mathbb{Z}} S_i$

has additive subgroups S_i such that $S_i S_j \subseteq S_{i+j}$.

Thus S_0 is a subring each S_i is an S_0-module.
S_i is <u>i^{th} component</u>.
$s \in S$ is homogeneous of degree i if $s \in S$.

A graded ideal I of S is an ideal of the form $\bigoplus I_i$ with $I_i \subset S_i$.

Note that such a graded ideal is f.g. as an ideal I. Then there is a finite generating set of homogeneous elements.

A <u>graded S-module</u> V is of the form $\bigoplus V_j$ such that $S_i V_j \subset V_{i+j}$.

Similarly for ~~the~~ positive graded rings / ideals or <u>negative</u> graded rings.

Note that a negatively graded ring may after renumbering be treated as a positive graded ring e.g. the associated graded ring of an I-adic filtration may be treated after renumbering as a positively graded ring.

[Notation: Using S again for graded ring, don't get confused !!]

Suppose $S = \bigoplus_{i \geq 0} S_i$ is (commutative) Noetherian generated by S_0 and homogeneous elements $x_1, \ldots, x_n$ of degree $k_1, \ldots, k_n$ respectively.

Let λ be an additive integral valued function on f.g. S_0-modules. (i.e. if $0 \longrightarrow U_1 \longrightarrow U_2 \longrightarrow U_3 \longrightarrow 0$ is a short exact sequence of f.g. S_0-modules).

then: $\lambda(U_2) = \lambda(U_1) + \lambda(U_3)$.

Examples: (1) If $S_0 = k$ field
$\lambda = k$-vector space dimension.

(2) If S_0 local, Artinian with maximal ideal P
Then for a f.g. S_0-module U there is a chain:

$$(0) = U_0 \subsetneq U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_n = U.$$

with $U_i / U_{i-1} \cong S_0/P$ for each i.

The number of factors is the composition length of U. and $\lambda(U) =$ comp. length of U is an additive function
(check: this is independent of the choice of chain (*))

(6.7) Def^n: The Poincaré series of $V = \bigoplus_{i \geq 0} V_i$,

a f.g. S-module $P(V, t) = \sum \lambda(V_i)\, t^i \in \mathbb{Z}[[t]]$

power series with integral coefficients.

(6.8) Hilbert-Serre Theorem

In the above situation:

$P(V,t)$ is a rational function of the form:

$$\frac{f(t)}{\prod_{j=1}^{n}(1 - t^{k_j})}$$

where $f(t) \in \mathbb{Z}[t]$ polynomial

k_j = degree of generator x_j.