Algebra

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Last time: Had my definitions without proof.
Now we'll prove stuff!

Recall: Hilbert - Serre theorem
$S = \bigoplus_{i \geq 0} S_i$ positively graded, Noetherian

V f.g. graded S-module

S generated by S_0 and $x_1, \dots, x_m$ of degree $k_1, \dots, k_m$. λ f.g. S_0-module $\longrightarrow \mathbb{Z}$ additive.

$$P(V,t) = \sum \lambda(V_i) t^i$$

Then: $$P(V,t) = \frac{f(t)}{\prod_j (1-t^{k_j})}$$ $f(t) \in \mathbb{Z}[t]$, k_j degrees of generators

$\longrightarrow$

(6.9) Corollary: If each $k_1 = \dots = k_m = 1$, then for large enough i:
$$\lambda(V_i) = \phi(i) \quad \text{for } \phi(t) \in \mathbb{Q}[t]$$
of degree $d-1$.
where d is the degree of the pole at 1 of $P(V,t)$

Moreover, $\sum_{j=0}^{i} \lambda(V_j) = \psi(i)$ for some $\psi(t) \in \mathbb{Q}[t]$ of degree d.

(6.10) Def^n: $\phi(t)$ Hilbert polynomial
$\psi(t)$ Samuel polynomial

(6.11) Defⁿ: We can use this d to give another concept of dimension $d(V)$ = degree of $\chi(t)$.
$d(S)$ = degree of $\chi(t)$ taking $V = S$.

Proof of Hilbert - Serre (6.8)

By induction on the number m of generators x_i.

$m = 0$: $S = S_0$ and V is a f.g S_0-module.
$S_0 \; V_i = 0$ for large enough i and
$P(V,t)$ is a polynomial.

$m > 0$: Assume true for $m-1$ generators.
Multiplication by x_m maps $V_i \xrightarrow{x_m} V_{i+k_m}$

and so we get an exact sequence:

$$(*) \quad 0 \longrightarrow K_i \longrightarrow V_i \xrightarrow{x_m} V_{i+k_m} \longrightarrow L_{i+k_m} \longrightarrow 0$$

K_i = kernel of the map 'mult by x_m'.
L_i = cokernel.

Let $K = \bigoplus K_i$, $L = \bigoplus L_i$

K is a graded submodule of $V = \bigoplus V_i$.
and hence a f.g. S-module (S is Noetherian)
$L = V/x_m V$ is also a f.g. S-module.

Note that x_m is acting like 0 on K and L
and so they may be viewed as f.g
$S_0[x_1, \dots, x_{m-1}]$-modules.

Applying λ to $\circledast$:

We get $\lambda(K_i) - \lambda(V_i) + \lambda(V_{i+k_m}) - \lambda(L_{i+k_m}) = 0$

Multiply by t^{i+k_m} and sum over i:

$$t^{k_m} P(K, t) - t^{k_m} P(V, t) + P(V, t) - P(L, t) = g(t) \quad \text{where } g(t) \in \mathbb{Z}[t].$$

arising from first few terms.

Apply the inductive hypothesis to $P(K, t)$ and $P(L, t)$

Proof of Corollary (6.9)

Here $k_1 = \dots = k_m = 1$.

$$P(V, t) = \frac{f(t)}{(1-t)^d} \qquad \begin{cases} f(t) \in \mathbb{Z}[t] \\ f(1) \neq 0. \end{cases}$$

Want power series for $(1-t)^{-d}$

Since $(1-t)^{-1} = 1 + t + t^2 + \dots$

Repeated differentiation gives:

$$(1-t)^{-d} = \sum \binom{d+i-1}{d-1} t^i$$

binomial coefficient.

Let $f(t) = a_0 + a_1 t + \dots + a_s t^s$ say.

Then:
$$\lambda(V_i) = a_0 \binom{d+i-1}{d-1} + a_1 \binom{d+i-2}{d-1} + \dots + a_s \binom{d+i-s-1}{d-1} \qquad (\dagger)$$

where $\binom{r}{d-1} = 0$ for $r < d-1$.

The RHS of $(\dagger)$ can be rearranged to give $\phi(i)$ for $\phi(t) \in \mathbb{Q}[t]$ valid for $d+i-s-1 \geq d-1$

$$\phi(t) = \frac{f(1)}{(d-1)!} t^{d-1} + \text{lower degree terms.}$$

Note: $f(1) \neq 0$ and so degree of $\phi(t)$ is $d-1$.

Using $(\dagger)$ we can produce an expression for $\sum_{j \leq i} \lambda(V_j)$:

$$\sum_{0}^{i} \binom{d+j-1}{d-1} = \binom{d+i}{d}$$

using $\binom{m}{n} = \binom{m-1}{n-1} + \binom{m-1}{n}$. (by Pascal's triangle).

and so: $$\sum_{j \leq i} \lambda(V_j) = a_0 \binom{d+i}{d} + a_1 \binom{d+i-1}{d} + \dots$$
$$\dots + a_s \binom{d+i-s}{d}$$

This is $\chi(i)$ for $\chi(t) \in \mathbb{Q}[t]$ of degree d.

<u>Example:</u> Take $S = k[X_1, \dots, X_m]$ polynomial algebra, k field, graded by total degree.
i^{th} component is the span of monomials of total degree i.

No of monomials of total degree i is $\binom{i+m-1}{m-1}$ for all $i \geq 0$.

Thus: $\phi(t) = \frac{1}{(m-1)!} (t+m-1)(t+m-2) \dots (t+1)$

product

This is Hilbert polynomial, degree is $m-1$.
Then $d(S) = m$ (w.r.t this grading).

Our aim is to apply this to P-adic filtrations of Noetherian rings R where P is a maximal ideal.

The P-adic filtration P^j – j^{th} term of filtration. R positive terms.

$$gr\, R = \bigoplus R_i / R_{i-1} \quad (i \in \mathbb{Z})$$
$$= \bigoplus P^j / P^{j+1} \leftarrow j^{th} \text{ component negatively graded ring.}$$

Renumber so that this is positively graded, and we can apply Hilbert-Serre. Note that $R/P = K$ field since P maximal $= S_0$.

$S_1 = P/P^2$. S_0 and S_1 generate $S = gr\, R$
$\lambda = \dim_K$ since P^i/P^{i+1} is a R/P-vector space.

So the corollary applies, and we have Hilbert and Samuel polynomials.

$$\phi(i) = \dim_K (P^i/P^{i+1}) \text{ for large enough } i.$$

$$\psi(i) = \sum_{j \leq i} \dim_K (P^j/P^{j+1}) = \text{composition length of } R/P^{i+1}.$$

(6.12) Def^n: $d_P(R) = d(gr\, R) = $ degree of $\psi(t)$
$= 1 + $ degree of $\phi(t)$

Another dimension of R - note it depends on choice of maximal ideals P.

<u>Unproved remarks</u> (not proved here, but someone else has proved them). ☺.

① For a Noetherian local ring, we have a unique maximal ideal P, and so can suppress the 'p' $d(R) = \dim R$ when / moreover R is an integral domain.

② For general Noetherian R, $d_p(R) = d(R_p)$ and so for integral domains:
$$d_p(R) = d(R_p) = \dim(R_p) = ht(P).$$

③ From example sheet 3 you know that for f.g. k-algebras R which are integral domains, all maximal ideals are of height $= \dim R$, and so we deduce $d_p(R) = \dim R$. for all maximal ideals P.

④ For a f.g. R-module M and max ideal P of R we can consider the P-adic filtration of M. $\{P^i M\}$. Define $d(M) = d(\mathrm{gr}\, M)$.

So we have a dimension for M.

[Note: There might be other filtrations around...!]

If $N \subseteq M$, there is the filtration induced on N from P-adic filtration of M
$(N \cap P^i M)$.

and we can again form ass. graded-module.

Reassuringly, the value of $d(N)$ obtained from this filtration and from the P-adic filtration of N are the same.

(Proof relies on Artin-Rees lemma - see ex. sheet 3)