Algebra 26/11/2019

§7. Homological Algebra

Book: C. Weibel
(Last time: We did Hilbert, Serre, applied p-adic filtration,

R commutative with a 1, but most of time doesn't need to be.
(just be careful with tensor product in non-commutative)

We've already seen that if we apply $N \otimes_R -$ to a short exact sequence, it may not remain exact (1.26) defined flat modules N to be those where exactness is preserved for all short exact sequences.

We also consider $\mathrm{Hom}_R(N, -)$ and $\mathrm{Hom}_R(-, N)$

If $0 \to M_1 \to M \to M_2 \to 0$ is exact we get $0 \to \mathrm{Hom}(N, M_1) \to \mathrm{Hom}(N, M) \to \mathrm{Hom}(N, M_2) \to 0$

and: $0 \to \mathrm{Hom}(M_2, N) \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M_1, N) \to 0$.

but these need not be exact. We do have exactness at left and middle terms, but not necessarily on right

e.g. If $N = \mathbb{Z}/2\mathbb{Z}$, $R = \mathbb{Z}$

$$0 \to 2\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

We get non-exactness.

(7.1) <u>Def^n</u>: A module P is <u>projective</u> if for any map $\phi : P \longrightarrow M_2$, then there is a map $P \longrightarrow M$, which composes with $M \longrightarrow M_2$ to give ϕ

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "\phi"] & \\ M \arrow[r] & M_2 \arrow[r] & 0 \end{tikzcd}$$

equivalently, $\mathrm{Hom}(P, -)$ preserves exactness of short exact sequences.

(7.2) <u>Def^n</u>: A module E is <u>injective</u> if $\mathrm{Hom}(-, E)$ preserves exactness of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\phi"'] & M \arrow[dl, dashed] \\ & E & \end{tikzcd}$$

<u>Examples</u>: ① Free modules are projective.
② the fraction field of an integral domain R is an injective R-module.

(7.3) <u>Lemma</u>: For an R-module P, the following are equivalent:

① P is projective.
② $\mathrm{Hom}(P, -)$ preserves exactness of short exact sequences.

③ If $\epsilon : M \longrightarrow P$ is surjective, then there exists $\beta : P \longrightarrow M$ with $\epsilon\beta = \text{identity}$.

④ P is a direct summand of every module of which it is a quotient.

⑤ P is a direct summand of a free module.

Proof: ① $\Rightarrow$ ② ✓

② $\Rightarrow$ ③

We have a short exact sequence

$$0 \longrightarrow \ker \varepsilon \longrightarrow M \xrightarrow{\varepsilon} P \longrightarrow 0$$

Then: $0 \longrightarrow \mathrm{Hom}(P, \ker \varepsilon) \longrightarrow \mathrm{Hom}(P, M) \longrightarrow \mathrm{Hom}(P, P) \longrightarrow 0$ is exact and so $\exists \beta : P \longrightarrow M$ such that $\varepsilon \circ \beta = id$.

③ $\Rightarrow$ ④:

$P = M/M_1$ and so we have

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow P \longrightarrow 0 \quad \text{exact.}$$

Apply ③: $\exists \beta : P \longrightarrow M$ such that $\alpha\beta = id$.
Therefore P is a direct summand of M.

④ $\Rightarrow$ ⑤:

Given P, we can define a free module on e_{x_λ} where $\{x_\lambda\}$ is a generating set of P.

$$\begin{aligned} F &\longrightarrow P \\ e_{x_\lambda} &\longmapsto x_\lambda \end{aligned} \quad \text{gives } R\text{-module map.}$$

Apply ④ to see that P is a direct summand of F

⑤ $\Rightarrow$ ①: By ⑤ $\underset{\text{free}}{F} = P \oplus Q$

free modules are projective and Hom behaves well w.r.t. direct summands and so P is projective.

Remarks:

① Given a PID, every f.g. projective is free (using structure theorem of f.g. modules over PIDs)

② There is a similar result giving equivalent statements for injective modules (ex. sheet)

③ Projective modules, being direct summands of free modules, are flat.

(7.4) Defn: A projective presentation of M is a short exact sequence:

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P projective. It is a free presentation if P is free.

Remark: The proof ④ $\Rightarrow$ ⑤ shows how to produce a free presentation of a module.

(7.5) (Defn) Given a projective presentation of M.
(i) Apply $N \otimes -$ to get:
$$N \otimes K \longrightarrow N \otimes P \longrightarrow N \otimes M \longrightarrow 0.$$

Define $\mathrm{Tor}^R(N, M) = \ker(N \otimes K \longrightarrow N \otimes P)$

(Note, we don't necessarily have kernel is 0, thus recording what the kernel is through Tor).

(ii) Apply $\mathrm{Hom}_R(-, N)$ to get:
$$0 \longrightarrow \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(P, N) \longrightarrow \mathrm{Hom}(K, N).$$

Define: $\mathrm{Ext}_R(M, N) = \mathrm{coker}(\mathrm{Hom}(P, N) \longrightarrow \mathrm{Hom}(K, N))$.

Thus, if ① N is flat, $\mathrm{Tor}^R(N, M) = 0 \quad \forall M$.
② E is injective $\mathrm{Ext}_R(M, E) = 0 \quad \forall M$.
③ If P is projective, then,
$$\mathrm{Ext}_R(P, N) = 0 \quad \forall N.$$

(If we have a projective presentation of P
$$(*) \quad 0 \longrightarrow K \longrightarrow P' \longrightarrow P \longrightarrow 0$$

But from (7.3), we know P is a direct summand of P', and so if we apply $\mathrm{Hom}(-, N)$ to $(*)$ we still have a short exact sequence and so we get $\mathrm{Ext}_R(P, N) = 0$.

Remark: ① Tor and Ext are independent of choice of projective presentation. (ex sheet)

Now: Tor^R superscript R (to generalise Tor_i^R)
Ext_R subscript R (to generalise Ext^i_R)

② One may take a projective presentation of N and apply $- \otimes_R M$ to it. The analogous kernel is isomorphic to $\mathrm{Tor}(N, M)$

③ Similarly, we could have taken a short exact sequence: $0 \longrightarrow N \longrightarrow \underset{\text{injective}}{E} \longrightarrow L \longrightarrow 0$

and consider $\mathrm{coker}\,(\mathrm{Hom}(M, E) \longrightarrow \mathrm{Hom}(M, L))$.

This is isomorphic to $\mathrm{Ext}(M, N)$

[can check these in own time, get same abelian groups by different approaches]

<u>Example</u>: $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}$-module has a free presentation:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Apply: $\mathbb{Z}/2\mathbb{Z} \otimes -$

$$\mathrm{Tor}^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \ker\big(\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}\big)$$

induced by mult. by 2

<u>zero map</u>

$$\cong \mathbb{Z}/2\mathbb{Z}$$

$\longrightarrow$

Now, let's apply $\text{Hom}_{\mathbb{Z}}(-, N)$ for any N $\mathbb{Z}$-module.

$$\text{Ext}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, N) = \text{coker}\left(\text{Hom}_{\mathbb{Z}}(\mathbb{Z}, N) \longrightarrow \text{Hom}(\mathbb{Z}, N)\right)$$

induced by mult. by 2!

$$= \text{coker}\left(N \xrightarrow[\text{mult. by } 2]{} N\right)$$

$$= N/2N$$