Algebra 28/11/2019.

Recall last time:

$$0 \longrightarrow \underset{}{\overset{\text{syzygy.}}{K}} \longrightarrow \overset{\text{projective}}{P} \longrightarrow M \longrightarrow 0$$

<u>projective presentation</u>.

Apply $N \otimes -$
Apply $\mathrm{Hom}(-, N)$

(observe that when we apply functors like the above, SES <u>don't</u> necessarily remain exact.)

$\mathrm{Tor}(N, M) = \ker(N \otimes K \longrightarrow N \otimes P)$
$\mathrm{Ext}(M, N) = \mathrm{coker}(\mathrm{Hom}(P, N) \longrightarrow \mathrm{Hom}(K, N))$.

K referred to as <u>syzygy module</u>.

<u>Remarks</u>: ① Ext can be defined alternatively using equivalence classes of extensions of M by N:

$$0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0.$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & X_1 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & N & \longrightarrow & X_2 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where θ R-module map.

Those two extensions are <u>equivalent</u> if such θ exists. θ necessarily has to be an isomorphism.

$\mathrm{Ext}(M, N) \cong \{$eq. classes of extensions of M by $N\}$

ex. sheet.

If $M = \mathbb{Z}/2\mathbb{Z}$, $N = \mathbb{Z}/2\mathbb{Z}$, $R = \mathbb{Z}$.

$\mathrm{Ext}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$

$\exists$ 2 eq. classes of extensions

$X = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$X = \mathbb{Z}/4\mathbb{Z}$.

One can define a sum of extensions – Baer sum.

So Ext terminology is entirely reasonable! ☺

Zero element of Ext $\longleftrightarrow$ direct sum

② The name Tor is a bit less clear.
It's from the case where $R = \mathbb{Z}$.
If A is an Abelian group,
the torsion subgroup $\cong \mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, A)$.

(Tor terminology not as well chosen...)

③ When R is commutative, $N \otimes_R M$ is an R-module
$\mathrm{Hom}_R(M, N)$ is an R-module.

and so $\mathrm{Tor}(N, M)$ and $\mathrm{Ext}(M, N)$ are R-modules. In general, they are additive groups but not necessarily R-modules.

Example: ① $R = k[X]$

$$0 \longrightarrow k[X] \longrightarrow k[X] \longrightarrow k \longrightarrow 0$$

[principal ideal (X)]

$g(x) \longmapsto Xg(x) \in (X)$

trivial $k[X]$-module where X acts like 0.

② $R = k[X, Y]$

$$0 \longrightarrow K \longrightarrow k[X,Y] \longrightarrow k \longrightarrow 0$$
$$f(X,Y) \longmapsto f(0,0)$$

K = ideal gen by X and Y.

$$0 \longrightarrow \overset{F_2}{k[X,Y]} \longrightarrow \overset{F_1}{k[X,Y] \oplus k[X,Y]} \longrightarrow K \longrightarrow 0$$

$$(g(X,Y), h(X,Y)) \longmapsto Xg(X,Y) + Yh(X,Y)$$

$$f(X,Y) \longmapsto (Yf(X,Y), -Xf(X,Y))$$

ideal gen. by X and Y.

Putting these together, we get exact sequence.

$$0 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow k \longrightarrow 0.$$

(7.6) Def^n: A projective resolution of an R-module M is an exact sequence of the form:

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

with P_i projective. It is a free resolution if all P_i are free modules.

Remark: ① We've just constructed a free resolution of k when $R = k[X, Y]$.

② If R Noetherian, and M is a f.g. R-module, then we saw how to construct a free presentation of M using a generating set.

This can be taken to be *finite*, and K syzygy module is then a f.g. R-module.

Repeating for K etc. gives a free resolution for M where all P_i are free modules of finite rank.

(7.7) Def^n: The *Koszul complex* gives a free resolution of the trivial R-module k when $R = k[X_1, \dots, X_n]$.

Define F_i to be free R-module on a basis $\{e_{j_1,\dots,j_i}\}$ where $j_1,\dots,j_i$ is a subset of $\{1,\dots,n\}$ of size i.

We define R-module map

$$F_i \longrightarrow F_{i-1}$$

$$e_{j_1,\dots,j_i} \longmapsto \sum_{l=1}^{i} (-1)^{l-1} X_{j_l} \, e_{j_1,\dots,j_{l-1},j_{l+1},\dots,j_i}$$

j_l is missing

size $i-1$.

Note that this is what we did in the case of $k[X]$ and $k[X,Y]$. (didn't have to write hed for $k[X]$, although $k[X,Y]$ more work)

This complex gives a free resolution — it is exact.

There are two approaches to defining higher Tor and Ext. Firstly, by iterating the previous definition.

(7.8) Def^n: $\mathrm{Tor}_0(N, M) = N \otimes M$
$\mathrm{Tor}_1(N, M) = \mathrm{Tor}(N, M)$

For $i \geq 2$, $\mathrm{Tor}_i(N, M) = \mathrm{Tor}_{i-1}(N, K)$, where K is syzygy module in a projective presentation for M.

(7.9) Def^n: $\mathrm{Ext}^0(M, N) = \mathrm{Hom}(M, N)$
$\mathrm{Ext}^1(M, N) = \mathrm{Ext}(M, N)$

For $i \geq 2$, $\mathrm{Ext}^i(M, N) = \mathrm{Ext}^{i-1}(K, N)$
where K is the syzygy module as before.

Alternatively, putting the presentations for syzygy module K_i together, we get projective resolution for M

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Apply $N \otimes$ — to get
$$\longrightarrow N \otimes P_i \xrightarrow{\theta_i} \cdots \longrightarrow N \otimes P_0 \longrightarrow N \otimes M \longrightarrow 0.$$

but this need not be exact but $\mathrm{Im}\,\theta_i \subseteq \ker \theta_{i-1}$.
and we can consider the homology groups $\ker\theta_{i-1}/\mathrm{Im}\,\theta_i$ and these are the $\mathrm{Tor}^i(N, M)$ for $i \geq 1$.

Similarly, applying $\mathrm{Hom}(—, N)$ to our projective resolution of M gives

$$0 \longrightarrow \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(P_0, N) \longrightarrow \cdots$$

and higher Ext groups arise as the cohomology groups of this.

Remarks: ① These are all independent of choice of presentations / resolutions.

(7.10) Lemma: The following are equivalent:

① $\mathrm{Ext}^{n+1}(M, N) = 0$ for all R-modules N.
② M has a projective resolution of length n.

$$0 \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

Proof: By induction:
$n=1$: $\mathrm{Ext}^2(M, N) = 0 \quad \forall N$
$\iff \mathrm{Ext}(K, N) = 0 \quad \forall N$
where K is syzygy module in a proj. presentation for M.

$\iff K$ is projective.
$\iff M$ has a proj. resolution as desired.

$\mathrm{Ext}^{n+1}(M, N) = 0 \ \forall N \iff \mathrm{Ext}^n(K, N) = 0 \ \forall n$
$\iff K$ has a projective resolution of length $n-1$.
$\iff M$ has desired resolution.

//