Algebra

30/11/2019

Had 2 lectures on homological things:

- Saw free and projective presentations.
- Syzygy was kernel of map.
- Last lecture, can build up resolution for our module.
- For some easy modules, things become easier to understand.

(7.11) Def^n: The projective dimension of M is n if $\mathrm{Ext}^{n+1}(M, N) = 0 \ \forall N$ but $\exists N$ such that $\mathrm{Ext}^n(M, N) \neq 0$.

Example: Koszul complex gave a free resolution of the trivial module k and we can deduce proj. dim $(k) = n$.
$R = k[X_1, \dots, X_n]$

The global dimension of R is the sup of all the proj. dimensions for f.g. R-modules M.

Examples: ① gl. dim $k = 0$ since all f.g.-modules are free.

② gl. dim $R = 1$ if R is a PID, which isn't a field.
(consider free presentation of a f.g. R-module then the syzygy module must be free).
e.g. $k[X]$.

③ gl. dim $k[X_1, \dots, X_n] = n$
(we've just seen that proj. dim $|k| = n$)

not proved here, but we will prove the following:

(7.12) Hilbert's syzygy theorem:

Let k be a field and $R = k[X_1, \ldots, X_n]$ considered as a graded algebra using total degree.
Let M be a f.g. graded R-module.
Then there is a free resolution of M of length $\leq n$.

Proof: relies on some of the properties of Tor that we were taking on trust.

(7.7) Koszul complex gives a free resolution of trivial module k of length n.

We'll consider $\mathrm{Tor}_i(k, M)$ in two different ways, by either taking a resolution for k (Koszul complex) or taking a resolution for M.

Either apply $- \otimes M$ to Koszul complex and consider homology groups, or $k \otimes -$ to a projective resolution for M.

For M we want to pick a minimal free resolution — at each stage we pick a minimal generating set in turn for each syzygy module.

With a minimal resolution, when we tensor with the trivial module:

$$\ldots \longrightarrow k \otimes F_i \longrightarrow k \otimes F_{i-1} \longrightarrow \ldots \longrightarrow k \otimes F_0 \longrightarrow k \otimes M.$$

where:

$$\ldots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \ldots \longrightarrow F_0 \longrightarrow M$$

is the minimal resolution.

all the maps $k \otimes F_i \to k \otimes F_{i-1}$ are zero maps

(zero by minimality & graded modules
If non-zero, then could lift to get direct summand, contradicts minimality)

The homology groups are therefore k-vector spaces of dimension = rank of the free modules.

However, we know that $\mathrm{Tor}_i(k, M) = 0$ for $i \geqslant n+1$ by looking at Koszul complex.

So we deduce that the F_i are of rank 0 for $i \geqslant n+1$. So our minimal resolution for M is of length $\leq n$.

Hochschild (co-) homology

This is the (co-) homology theory for R-R bimodules where R is a k-algebra. An R-R bimodule M (R acting on left and right and two actions commute) may be viewed as a right $R \otimes_k R^{op}$-module.

R^{op} is a k-algebra with the same elements r as R but with multiplication reversed:

$$r \cdot s = sr$$

multiplication in R^{op}

multiplication in R.

$$rms = m(s \otimes r) \qquad \begin{matrix} s \in R \\ r \in R^{op} \end{matrix}$$

(So can think instead of bimodules as one-sided modules, but then must make ring bigger $R \otimes R^{op}$.)

Similarly M can be regarded as a left $R \otimes R^{op}$-module.

Example: ① If R is commutative, then $R^{op} = R$.

② If $R = kG$ (group algebra), then $R^{op} \cong R$

by $\sum \lambda_g g \longrightarrow \sum \lambda_g g^{-1}$

Note that: ① R itself is an R-R bimodule.
② $R \otimes_k R$ is an R-R bimodule. Regard it as a right $R \otimes R^{op}$-module, then it is free generated by $1 \otimes 1$. Thus regard $R \otimes_k R$ as the free R-R bimodule of rank 1.

③ We have a free presentation of R as a bimodule

$$0 \longrightarrow \ker \mu \longrightarrow R \otimes_k R \xrightarrow{\mu} R \longrightarrow 0$$

$$r \otimes s \longmapsto rs$$

multiplication in R. This is a bimodule map.

(7.13) Def^n: R is a k-separable algebra if R is a projective R-R bimodule (or equivalently projective right $R \otimes R^{op}$-module)

and thus R can be regarded as a direct summand of $R \otimes_k R$ (embedded)

Note that a finite field extension K of k is k - ~~separable~~ separable $\Longleftrightarrow$ it is a separable field extension of k.

(On ex. sheet, we'll get this next time).

Remark: Hochschild (co-) homology is 'good', because it applies to any k-algebra R, not just to ones where there is a canonical map $R \to k$.

Also the multiplication in the algebra R is encoded clearly in the resolution of R.

(7.14) Def^n: The Hochschild chain complex gives a free bimodule resolution of R.

$$\cdots\cdots \xrightarrow{d} R\otimes R\otimes R\otimes R \xrightarrow{d} \underset{\text{degree }1}{R\otimes R\otimes R} \longrightarrow \underset{\text{degree }0}{R\otimes R} \xrightarrow{\mu} R \longrightarrow 0$$

μ: multiplication in R

$$d(\underbrace{r_0\otimes \cdots\cdots \otimes r_{n+1}}_{n+2 \text{ terms}}) = \sum_{i=0}^{n} (-1)^i \underbrace{r_0 \otimes \cdots \otimes r_{i-1} \otimes r_i r_{i+1} \otimes \cdots \cdots \otimes r_{n+1}}_{n+1 \text{ terms}}.$$

This is exact.

(7.15) Def^n: Given an $R-R$ bimodule M

Hochschild homology

$$HH_i(R,M) = \operatorname{Tor}_i^{R-R}(R,M).$$
$$= \operatorname{Tor}_i^{R\otimes R^{op}}(R,M) \quad \text{say}$$

Observe that we need to regard R and its resolution as right $R\otimes R^{op}$-modules and M as a left $R\otimes R^{op}$-module so that we can tensor

$R \otimes_{R \otimes R^{op}} M.$

<u>Hochschild cohomology groups</u>

$$HH^i(R, M) = \mathrm{Ext}^i_{R\text{-}R}(R, M) = \mathrm{Ext}^i_{R \otimes R^{op}}(R, M).$$

(Will do remarks next time)