Algebra

03/12/2019

Recall: · Last time we started thinking about bimodules.
· Hochschild (co)-homology.

We considered $R-R$ bimodules.

$$0 \longrightarrow K \longrightarrow R \otimes R \longrightarrow R \longrightarrow 0$$

free presentation of $R-R$ bimodules.

An $R-R$ bimodule is a <u>right</u> $R \otimes R^{op}$-module and a <u>left</u> $R \otimes R^{op}$-module.

$HH_i(R, M) = \mathrm{Tor}_i^{R-R}(R, M)$ — Hochschild homology

$HH^i(R, M) = \mathrm{Ext}^i_{R-R}(R, M)$ — <u>co</u>homology

(easier, since $R \otimes R^{op}$ doesn't arise, can just think about Homs)

In particular, $HH^0(R, M) = \mathrm{Hom}_{R-R}(R, M)$

$$\cong \{ m \in M : rm = mr \ \forall r \in R \}.$$

$\therefore HH^0(R, R) = Z(R)$ centre of R.

$HH_0(R, M) \cong M / \langle rm - mr : m \in M, r \in R \rangle$

$HH_0(R, R) \cong R/[R, R]$ $\quad [r,s] = rs - sr$ Lie bracket.

(7.16) (Def^n) The (Hochschild cohomological) dimension Dim R of R (capital D)

$$= \sup\{n : HH^n(R, M) \neq 0 \text{ for some bimodule } M\}.$$

<u>Remark:</u> Last time we defined k-separable algebras to be those R s.t. R is a projective bimodule.

This is equivalent to saying $HH^i(R, M) = 0$ for $i \geq 1$ and all M. These are those R s.t. R embeds in $R \otimes R$ as a direct summand.

Now consider higher cohomology groups.

Note that $\mathrm{Hom}_{R\text{-}R}(\underbrace{R\otimes \cdots \otimes R}_{n+2}, M) \cong \mathrm{Hom}_k(\underbrace{R\otimes \cdots \otimes R}_{n \text{ times}}, M)$

e.g. $\mathrm{Hom}_{R\text{-}R}(R\otimes R, M) \cong \mathrm{Hom}_k(\underset{\text{spanned by } 1\otimes 1}{k}, M) \cong M.$

(7.17) <u>Def^n:</u> The Hochschild cochain complex using k-linear maps:

$$M \cong \mathrm{Hom}_k(k, M) \xrightarrow{\delta_0} \mathrm{Hom}_k(R, M) \xrightarrow{\delta_1} \mathrm{Hom}_k(R\otimes R, M) \xrightarrow{\delta_2} \cdots$$

$$(\delta_0 f)(r) = r f(1) - f(1) r.$$

$$(\delta_1 f)(r_1 \otimes r_2) = r_1 f(r_2) - f(r_1 r_2) + f(r_1) r_2$$

$$(\delta_2 f)(r_1 \otimes r_2 \otimes r_3) = r_1 f(r_2 \otimes r_3) - f(r_1 r_2 \otimes r_3) + f(r_1 \otimes r_2 r_3) - f(r_1 \otimes r_2) r_3.$$

For $HH^1(R, M)$ we're interested in $\ker\delta_1 / \operatorname{im}\delta_0$

(7.18) (Def^n) $\ker\delta_1 = \{ f \in \operatorname{Hom}_k(R, M) : f(r_1 r_2) = r_1 f(r_2) + f(r_1) r_2 \}$

Leibniz derivations from R to M.

$\operatorname{Der}(R, M)$.

$\operatorname{Im}\delta_0 = \{ f \in \operatorname{Hom}_k(R, M)$ of the form $r \longrightarrow rm - mr$ for some $m \in M \}$

inner derivation from R to M

$$HH^1(R, M) = \operatorname{Der}(R, M) / \underset{\substack{\uparrow \\ \text{inner derivation.}}}{\operatorname{Innder}(R, M)}$$

Setting $M = R$, we get the inner derivations of R.

Example: $R = k[X]$ char $k = 0$.

Then $\operatorname{Der} k[X] = \{ p(x) \frac{\partial}{\partial x} \in \operatorname{End}_k(k[x]) \}$
$p(x) \in k[X]$.

When R is commutative as here:

$\operatorname{Innder} R = 0$. $HH^1(R, R) = \operatorname{Der} R$.

Note that $\operatorname{Der} R$ form a Lie Algebra with the Lie bracket inherited from $\operatorname{End}_k(R)$.

$[D_1, D_2] = D_1 D_2 - D_2 D_1 \leftarrow$ also a derivation.

The derivations are examples of differential operators.

(7.19) (Def^n) For a commutative algebra R we define differential operators inductively.

$$\mathcal{D}^0(R) = \{ D \in \mathrm{End}_k(R) : [r, D] = 0 \quad \forall r \in R \}.$$

multiplication by $r \in \mathrm{End}_k(R)$.

$$\mathcal{D}^1(R) = \{ D \in \mathrm{End}_k(R) : [r, D] \in \mathcal{D}^0(R) \}.$$

$$\mathcal{D}^2(R) = \{ D \in \mathrm{End}_k(R) : [r, D] \in \mathcal{D}^1(R) \}$$

etc...

We define $\mathcal{D}(R) = \bigcup \mathcal{D}^i(R)$

algebra of differential operators.

$\mathcal{D}(k[x]) = k[x ; \frac{\partial}{\partial x}]$ char $k = 0$

first Weyl algebra.

Exercise: Work out $\mathcal{D}(k[x])$ when char $k = p > 0$.

(this isn't the same, do get other diff operators for char $p > 0$).

Let's finish with a theorem! ☺.

(7.20) Hochschild - Kostant - Rosenberg (HKR Theorem)

Let $R = k[X_1, \ldots, X_n]$ (Think about $HH^i(R, R)$)

char $k = 0$.

$$HH^i(R,R) \cong \bigwedge \mathrm{Der}(R)$$

exterior algebra, graded algebra

0^{th} component $= R$
1^{st} component $= \mathrm{Der}(R)$
2^{nd} component $= \mathrm{Der}(R) \wedge \mathrm{Der}(R)$

$$V \wedge_R V = \frac{V \otimes_R V}{\langle v \otimes v : v \in V \rangle}$$

for an R-module V

i^{th} component is $\bigwedge^i \mathrm{Der}(R)$

$\mathrm{Der}(R) \wedge \mathrm{Der}(R) \wedge \mathrm{Der}(R)$ is 3^{rd}

$\wedge$ gives a product on the exterior algebra

There is also a Lie bracket. Schouten bracket
This is typical — on $HH^*(R,R)$ one has a product and a Lie bracket, and in fact on the Hochschild cochain complex there is additional algebraic structure

$\rightsquigarrow$ deformation of algebras (links to essay topic!)
In particular $HH^2(R,R)$ is important!

(First examples class: Tuesday of next term (before lectures start))