Algebra

22/10/2019

Q9 Better phrasing:
... if and only if each non-zero prime ideal contains a non-zero principal prime ideal.

Deadline for handing in work: Monday 10am

Recap: Defined tensor product
- Bilinear maps correspond to linear maps
- Take free module, and quotient out relations.
- Restriction / Extension of scalars.

Tensor product of maps

Given $\theta : M_1 \longrightarrow M_2$, $\phi : N_1 \longrightarrow N_2$ R-module maps.

then we take tensor product:

$$\theta \otimes \phi : M_1 \otimes N_1 \longrightarrow M_2 \otimes N_2$$
$$m_1 \otimes n_1 \longmapsto \theta(m_1) \otimes \phi(n_1)$$

R-module map.

(Note: The map: $M_1 \times N_1 \longrightarrow M_2 \otimes N_2$, $(m_1, n_1) \longmapsto \theta(m_1) \otimes \phi(n_1)$ is bilinear and so universality gives $\theta \otimes \phi$)

Given a short exact sequence:

$$0 \longrightarrow M_1 \xrightarrow{\theta} M \xrightarrow{\phi} M_2 \longrightarrow 0 \qquad (*)$$

We can form:

$$0 \longrightarrow N \otimes M_1 \xrightarrow{id \otimes \theta} N \otimes M \xrightarrow{id \otimes \phi} N \otimes M_2 \longrightarrow 0 \qquad (**)$$

where $id : N \longrightarrow N$ is the identity map.

We saw that when $N = S^{-1}R$ then localisation is <u>exact</u> (1.14).
and so $(**)$ is a short exact sequence.
($S^{-1}R \otimes_R M \cong S^{-1}M$ recall).

<u>However</u>, in general $(**)$ is not necessarily exact.

Example $R = \mathbb{Z}$.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$
$$n \longmapsto 2n$$

is a short exact sequence

Set $N = \mathbb{Z}/2\mathbb{Z}$ Note: $N \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$
$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$

So $(**)$ in this case:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow[id \otimes 2]{} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

with $id \otimes 2$ is the zero map.

We do <u>not</u> have exactness on the left-hand side. So $N \otimes_{\mathbb{Z}} -$ does not preserve exactness of short exact sequences. This is the starting point of <u>homology theory</u> ☺.

(1.26) <u>Defn</u>: N is a <u>flat</u> R-module if $N \otimes_R -$ preserves exactness of all short exact sequences.

<u>Example</u>: $S^{-1}R$ is a flat R-module for all multiplicatively closed subsets S of R.

In particular R itself is a flat R-module.

Or R^n is flat R-module (or direct summands).

(1.27) <u>Def^n</u>: 'tensor products of algebras'
Given $\phi: R \longrightarrow T$, a ring homomorphism
(and so T can be viewed as an R-module)

We say that T is an <u>R-algebra</u>.
Given two such maps: $\phi_i : R \longrightarrow T_i \quad (i=1,2)$
we can define their <u>tensor product over R</u>
by endowing $T_1 \otimes_R T_2$ with a product
$$(t_1 \otimes t_2)(t_1' \otimes t_2') = t_1 t_1' \otimes t_2 t_2'$$

(Remember not all elements of this form, but if we know how this multiplies, we know how others multiply.)

and
$$R \longrightarrow T_1 \otimes_R T_2$$
$$r \longmapsto \phi_1(r) \otimes 1 = 1 \otimes \phi_2(r) = r(1 \otimes 1)$$

<u>Check</u> well-defined.
$1 \otimes 1$ is the multiplicative identity.

<u>Examples</u> ① k field, $k[x_1] \otimes_k k[x_2] \cong k[x_1, x_2]$

② $\mathbb{Q}[x]/(x^2+1) \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}[x]/(x^2+1)$

③ $k[x_1]/(f(x_1)) \otimes_k k[x_2]/(g(x_2)) \cong k[x_1, x_2]/(f(x_1), g(x_2))$

So we've done <u>localisation</u> and <u>tensor products</u>. ☺

Chapter 2: Ideal Structure.

Throughout this chapter, R is commutative with a 1.

(2.1) Lemma: The set of nilpotent elements of a commutative ring R form an ideal $\mathcal{N}(R)$
NB! (else consider ring of matrices, this doesn't work)

and $R/\mathcal{N}(R)$ has no non-zero nilpotent elements.

Proof: Exercise: x nilpotent $\Rightarrow rx$ nilpotent
x, y nilpotent $\Rightarrow x+y$ nilpotent
(just use binomial theorem).

(2.2) Def^n: The ideal $\mathcal{N}(R)$ is the nilradical of R.
curly N

(2.3) Lemma: (Krull) $\mathcal{N}(R)$ is the intersection of all the prime ideals of R.

Proof: Let $I = \bigcap_{P \text{ prime}} P$

If $x \in R$ is nilpotent, then $x^m = 0$ for some m.
P prime $\Rightarrow x \in P$ (note $0 \in P$)
So $x \in \bigcap P = I$.

If x is not nilpotent. Then consider
$S = \{1, x, x^2, \dots\} \not\ni 0$. (as x not nilp

Localise $S^{-1}R = R_x$ (recall notation).
Note that R_x is not zero, since $0 \notin S$.

Take a maximal ideal of R_x (by Zorn's lemma)

By (1.16) (ii), this maximal ideal of R_x corresponds to a prime ideal P with $P \cap S = \emptyset$.
In particular $x \notin P$. Thus $x \notin \bigcap_{P \text{ prime}} P = I$.

Thus: $I = \mathcal{N}(R)$ $\square$

$\hookrightarrow$ curly N.

(Alternative proof: Consider all ideals that don't have x, and apply Zorn)

(2.4) Def^n: For an ideal I of R, it's radical $\sqrt{I} = \{x : x^n \in I \text{ for some } n\}$.

Note that: $\sqrt{I}/I = \mathcal{N}(R/I)$ $\leftarrow$ nilradical of R/I.

(2.5) Def^n: The Jacobson radical of R is the intersection of all the maximal ideals of R.

$\mathrm{Jac}(R)$ = Jacobson radical of R.

Thus $\mathcal{N}(R) = \bigcap_{P \text{ prime}} P \leq \bigcap_{P \text{ maximal}} P = \mathrm{Jac}(R)$

In general, we don't get equality.
e.g. If R local ring with unique maximal ideal P, then $\mathrm{Jac}(R) = P$.

E.g: $R = \{m/n \mid p \nmid n$ for some prime $p\}$ with unique max ideal $P = \{\frac{m}{n} \mid p \mid m, p \nmid n\}$.

But R is integral domain, thus any nilpotent is 0.

Thus: $\mathcal{N}(R) = \{0\} \neq \mathrm{Jac}(R) = P$

$\longrightarrow$

(In some contexts, local ring might not be well-behaved..)

For *finitely-generated algebras*, nilradical *is* same as Jacobson radical.