Algebra

24/10/2019

Reminders: Hand-in deadline 10am Monday
- Can hand-in anything, but Q8, 9 most substantial.

Last time: Did nilradical and Jacobson radical
$\bigcap$ primes $\qquad$ $\bigcap$ maximal

(2.6) Nakayama's Lemma
- Let M be a finitely generated R-module.
- Then $M = 0 \iff \operatorname{Jac}(R) M = M$

Proof: ($\Rightarrow$) Obvious ✓

($\Leftarrow$) Suppose $M \neq 0$. We consider the family of proper submodules. If $m_1, \dots, m_n$ is a generating set for M, these are the submodules that do not contain all of $m_1, \dots, m_n$. Zorn applies to this family
(family of proper submodules)

and so we have a maximal member, a maximal submodule M_1. Thus M/M_1 is a simple (or irreducible) module (i.e. any non-zero element generates M/M_1)
(only subs are 0 or whole thing)

Take $\bar{m}$ to be non-zero in M/M_1, $R\bar{m} \cong R/\operatorname{Ann}(\bar{m})$.
and $\operatorname{Ann}_R(\bar{m}) = Q$ is a maximal ideal of R.

So $QM \leq M_1 \lneq M$

But $\operatorname{Jac} R$ = intersection of all maximal ideals.
So $\operatorname{Jac}(R) M \leq QM \lneq M$!

Thus $\operatorname{Jac}(R) M \neq M$

$\square$

Ours is more general.

Remarks: ① This is not the normal proof you find in Atiyah, MacDonald.

② The same proof shows that for fin. gen. R-modules M,

$M = 0 \iff QM = M$ for all maximal ideals Q.

Exercise: Find an R and an R-module $M \neq 0$ but $QM = M$ for all maximal Q.

(M cannot be fin. gen.)

Nullstellensatz

The name is attached to a family of results.

(2.7) Theorem: (Weak Nullstellensatz).

Let k be a field, and T be a finitely generated k-algebra. Let Q be a maximal ideal. Then T/Q is a finite field extension of k.

In particular, if k is algebraically closed and $T = k[x_1, \ldots, x_n]$ polynomial algebra, then Q is of the form $(X_1 - a_1, \ldots, X_n - a_n)$ for some $(a_1, \ldots, a_n) \in k^n$.

We need two results:

(2.8) Lemma: (Artin, Tate)

Let $R \subseteq S \subseteq T$ be rings.

Suppose R is Noetherian and T is generated as a ring by R and $t_1, \ldots, t_n$, say.

Suppose moreover that T is finitely generated S-module.

Then S is generated as a ring by R and finitely many elements.

<u>Proof:</u> Let T be generated by $x_1, \dots, x_m$ as an S-module.

For each $t_i := \sum s_{ij} x_j$ for $s_{ij} \in S$ ①

$x_i x_j = \sum s_{ijk} x_k$ for $s_{ijk} \in S$. ②

Let S_0 be the subring of S generated by R and all the s_{ij}, s_{ijk}. Then $R \subseteq S_0 \subseteq S$.

Any element of T is a 'polynomial' in the t_i with coefficients in R!
Using ① and ② we get that each element of T is a linear combination of the x_j with coefficients in S_0. Thus T is a fin. gen. S_0-module.

S_0 is <u>Noetherian</u> as it's a fin. gen. R-algebra, and R is <u>Noetherian</u> (corollary of basis theorem)

Hence, T is a Noetherian S_0-module.
But S is an S_0-submodule of T, and hence is a fin. gen. S_0-module. But S_0 is generated as a ring by R and finitely many elements.

So S is gen as ring by R and finitely my elemts.

(2.9) <u>Proposition:</u> Let k be a field, and R be a fin. gen. k-algebra.
If R is a field, then it is a finite field extension of k.

Proof: Suppose R is generated by k and $x_1, \dots, x_n$ and is a field. If R is algebraic over k then it is a finite field extension.

If R is not algebraic, we reorder the x_i's so that $x_1, \dots, x_m$ are algebraically independent and $x_{m+1}, \dots, x_n$ are algebraically dependent on $F = k(x_1, \dots, x_m)$

(shift dependence to the right).

Hence, R is a finite field extension of F and hence a fin. gen. F-module (i.e. an F vector space)

Apply (2.8) to $k \subseteq F \subseteq R$. It follows that F is a fin. gen. k-algebra. Suppose F is generated by k and $q_1, \dots, q_t$ where $q_i = \frac{f_i}{g_i}$.

$f_i, g_i \in k[x_1, \dots, x_m]$ polynomial algebra $g_i \neq 0$.

There is a polynomial h which is prime to $g_1, \dots, g_t$ e.g. $\underbrace{g_1 \cdots g_t}_{\text{product}} + 1$.

Then $\frac{1}{h}$ is not in the ring generated by by k and the q_i. $\times$.

Proof of the weak Nullstellensatz:

Let Q be a maximal ideal of T, f.g. k-algebra

Set $R = T/Q$ and apply (2.9)

Thus R is a finite field extension of k.

Now, if $T = k[X_1, \dots, X_n]$ and k algebraically closed. Then $T/Q \cong k$ (using algebraic closure)

Set $\pi : T \longrightarrow k$ with $\ker \pi = Q$. But (quotient map)
$\ker \pi = (X_1 - \pi(X_1), \dots, X_n - \pi(X_n))$

(Note that $k[X_1, \dots, X_n] / (X_1 - \pi(X_1), \dots, X_n - \pi(X_n))$ is 1-dimensional.)

Hence, Q is of the form $(X_1 - a_1, \dots, X_n - a_n)$ with $(a_1, \dots, a_n) \in k^n$.

Now, let $k = \mathbb{C}$.
In the introduction, we met a bijection:

$$\left\{ \begin{matrix} \text{radical ideals} \\ \text{of } \mathbb{C}[X_1, \dots, X_n] \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{algebraic} \\ \text{subsets of } \mathbb{C}^n \end{matrix} \right\}$$

The weak Nullstellensatz, says that all the maximal ideals of the complex polynomial algebra are of the form $Q_{(a_1, \dots, a_n)} = (X_1 - a_1, \dots, X_n - a_n)$.

We can restate the bijection:

$$\left\{ \begin{matrix} \text{radical} \\ \text{ideals} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{algebraic} \\ \text{subsets} \end{matrix} \right\}$$

$$I \longmapsto \{(a_1, \dots, a_n) \in k^n, \ Q_{(a_1, \dots, a_n)} = I\}$$

$$\bigcap_{(a_1, \dots, a_n) \in \mathcal{S}} Q_{(a_1, \dots, a_n)} \longleftarrow\!\shortmid \ \mathcal{S}$$

(2.10) Theorem: Let k be a field.
R a fin. gen. k-algebra.
Then $\mathcal{N}(R) = \mathrm{Jac}(R)$.
Thus, if I is a radical ideal of $k[X_1, \dots, X_n]$ and $R = k[X_1, \dots, X_n]/I$, then the intersection of the maximal ideals of R is zero, and thus the intersection of the maximal ideals of $k[X_1, \dots, X_n]$ containing I is equal to I.

[Will prove on Saturday