Algebra 26/10/2019

(2.10) **Theorem:** Let k be a field. R fin gen. k-algebra. Then $\mathcal{N}(R) = \mathrm{Jac}(R)$.

We'll need the following for the proof:

(2.11) **Lemma:** Let k be a field, R be an integral domain which is finite dimensional as a k-vector space. Then R is a field.

Proof: Take $0 \neq r \in R$, Then $\theta_r : R \longrightarrow R$, $x \longmapsto rx$

is k-linear, and injective because R is integral dom.

By rank-nullity, θ_r is surjective, and so $1 \in \mathrm{Im}\,\theta_r$. Thus there is an x s.t. $rx = 1$. So r has an inverse.

As r any non-zero, Thus R is a field. $\square$

Proof of (2.10).
Let P be a prime ideal of R, $s \in R \setminus P$. We set $S = \{1, s, s^2, \dots\}$. Localise w.r.t. S. So we have $S^{-1}R$, where $\theta : R \longrightarrow S^{-1}R$ is the canonical map.

$S^{-1}R$ is f.g. as k-algebra since it is generated by $\theta(R)$ and $\frac{1}{s}$. $(s \notin P)$

Take a maximal ideal Q of $S^{-1}R$ containing $S^{-1}P$. By (2.9) $S^{-1}R/Q$ is a finite field extension of k. (weak Nullstellensatz).

By (1.16) Q corresponds to a prime ideal P_1 of R containing P_1 and not containing S.

Also, $P_1 = \{ r \mid r/1 \in Q \} = \theta^{-1}(Q)$.

θ induces an embedding: $R/P_1 \longrightarrow \boxed{S^{-1}R/Q}$

finite dimensional k-vector space $\rightarrow$

So R/P_1 is a finite dim. k-vector space.
By (2.11) R/P_1 is a field. Thus P_1 is a maximal ideal containing P but not containing S.

Thus: $\bigcap_{\substack{\text{maximal ideals} \\ \text{containing } P}} = P$

$\therefore \quad N(R) = \bigcap \left({\text{prime} \atop \text{ideals}}\ P \right) = \bigcap_{\substack{\text{maximal} \\ \text{ideals}}} = \mathrm{Jac}(R)$. $\square$

(Similar to Krull's proof, but used weak Nulls here to get $S^{-1}R/Q$ fin. dim.) ☺.

Minimal and associated primes.

(2.12) Lemma: If R is Noetherian, then every ideal I contains a power of its radical $\sqrt{I}$.
In particular $N(R)$ is nilpotent (i.e. $N(R)^t = 0$ for some t).
(stronger than just each element nilpotent)

Proof: Suppose $x_1, \dots, x_n$ generate $\sqrt{I}$ as an ideal. Then $x_i^{m_i} \in I$ for some m_i $(i = 1, \dots, n)$.

Let $m = \left(\sum (m_i - 1)\right) + 1$. Then $(\sqrt{I})^m \subseteq I$, since $(\sqrt{I})^m$ is generated by elements of the form $x_1^{r_1} \cdots x_n^{r_n}$ with $\sum r_i = m$. and at least one of the r_i has to be $\geq m_i$, and so all these products lie in I.

(2.13) Lemma: If R is Noetherian, a radical ideal is the intersection of finitely many primes

Proof: Suppose not, and I is a maximal member of the set of radical ~~sets~~ ideals which aren't an intersection of finitely many primes

Claim: I is prime.
Proof of claim: Suppose not. Then there are ideals J_1, J_2 with $I \geq J_1 J_2$ but $I \subsetneq J_1$, $I \subsetneq J_2$.

(note: alternative definition of primeness using ideals

[Note: $J_1 = \langle I, a \rangle$, $J_2 = \langle I, b \rangle$ $a \notin I$, $b \notin I$, but $ab \in I$].

Maximality of I forces $\sqrt{J_1}$ and $\sqrt{J_2}$ to be the intersections of finitely many primes.

$$\sqrt{J_1} = Q_1 \cap \cdots \cap Q_s \qquad Q_i, Q_j' \text{ primes}$$
$$\sqrt{J_2} = Q_1' \cap \cdots \cap Q_t'$$

Set $J = Q_1 \cap \cdots \cap Q_s \cap Q_1' \cap \cdots \cap Q_t' = \sqrt{J_1} \cap \sqrt{J_2}$
So $J^{m_1} \subseteq J_1$ and $J^{m_2} \subseteq J_2$ for some m_1, m_2 by (2.12).

by assumption

Hence, we have $J^{m_1+m_2} \subseteq J_1 J_2 \subseteq I$

But, I is radical, and so $J \subseteq I$.

However, all the Q_i, Q_j' contained I, and so $I \subseteq J$. Thus $I = J$.

which is contradiction, since I was supposed not to be an intersection of finitely many primes.
So I is prime, itself a contradiction.

So lemma established. $\square$

Now suppose I is an ideal of Noetherian R.
By (2.13), $\sqrt{I} = P_1 \cap \dots \cap P_n$ for some primes P_i. We may remove any P_i if it contains one of the others, and so we may assume $P_i \not\subseteq P_j$ for $i \neq j$. Note that if P is any prime containing I, then $\underbrace{P_1 \dots P_n}_{\text{product}} \subseteq P_1 \cap \dots \cap P_n = \sqrt{I}$

and $\sqrt{I} \subseteq P$ (as any prime containing I contains $\sqrt{I}$).

Thus, some $P_i \subseteq P$.

(2.14) **Def^n**: The minimal primes over I are the minimal members of the set of primes containing I.

What we have just produced for an ideal of a Noetherian ring is the set of minimal primes over I.

Let's record this as a lemma (already proved)

(2.15) Lemma: Let I be an ideal of Noetherian ring R. Then the set of minimal primes over I is finite, and their intersection is $\sqrt{I}$, and I contains some finite product of the minimal primes (perhaps with repetition).

(2.16) Def^n: Let M be a fin. gen. R-module, R Noetherian. A prime ideal P is an associated prime of M if $P = \mathrm{Ann}_R(m)$ for some $m \in M$.

Notation: $\mathrm{Ass}(M)$ = set of associated primes of M.

(2.17) Def^n: A submodule N of M is P-primary if $\mathrm{Ass}(M/N) = \{P\}$.

Remark: ① Suppress P if you can.
② There is an alternative definition of P-primary
– See next ex. sheet.

(Not enough to think about minimal primes, really do have to think about all associated primes.)