# Algebra

29/10/2019.

- This Friday 3-4pm MR13 Junior algebra seminar
- Local research students talk for 7, 8 mins about PhD research.

Last time: Met minimal primes, $\sqrt{I}$ intersection of finitely many primes, these are precisely minimal primes.

Our aim is to show that, for $R$ Noetherian, non-zero f.g. $R$-modules $M$. $\mathrm{Ass}(M)$ is non-empty and finite, and in the case of $M = R/I$, then $\mathrm{Ass}(R/I) \supseteq \{\text{minimal primes over } I\}$.

However, we don't necessarily have equality here.

Example: $R = k[X, Y]$

$Q = (X, Y) \geqslant P = (x)$

$I = PQ$

Then $\mathrm{Ass}(R/I) = \{P, Q\}$

but the only minimal prime over $I$ is $P$.

Note that $I$ is not primary

but that $I = (X^2, XY, Y^2) \cap (X)$.

$(X^2, XY, Y^2) = Q^2$ is $Q$-primary.

$(X)$ is $P$-primary

(Primary ideals don't have to be just powers of primes!)

This is an example of primary decomposition.

Primary decomposition (not proved here)
$R$ Noetherian, $M$ f.g. $R$-module, $N$ submodule of $M$. Then there exist $N_1, \dots, N_s$ with
$N_1 \cap \dots \cap N_s = N$ and $\mathrm{Ass}(M/N_i) = \{P_i\}$
for prime ideal $P_i$, all distinct.

(Thus $N_i$ is a $P_i$-primary submodule of $M$).

In particular, given an ideal $I$ of $R$, then
$I = J_1 \cap \dots \cap J_s$ for some $P_i$-primary ideals $J_i$. If one takes a minimal decomposition like this, then the $P_i$ appearing are precisely the associated primes of $I$.

In practice, one needs to know $\mathrm{Ass}(M)$ or $\mathrm{Ass}(R/I)$ rather than the primary decomposition. Instead you tend to consider localisations.

(2.18) Lemma: If $\mathrm{Ann}(M) = \{r : rm = 0 \ \forall m \in M\}$ (annihilator of whole module)
$= \bigcap_{m \in M} \mathrm{Ann}(m)$

for a f.g. $R$-module $M$, is a prime ideal $P$, then $P \in \mathrm{Ass}(M)$.

Proof: Let $m_1, \dots, m_s$ be a generating set for $M$, $I_j = \mathrm{Ann}(m_j)$. Then $\prod I_j$ annihilates each $m_j$ and so $\prod I_j \subseteq \mathrm{Ann}(M) = P$, by supposition.

Hence, some $I_j \subseteq P$ since $P$ is prime.
However, $I_j = \mathrm{Ann}(m_j) \supseteq \mathrm{Ann}(M) = P$.
So $P = I_j$ for some $j$ and so is the annihilator of an element

$P \in Ass(M)$.

(2.19) Lemma: 'Ass(M) for non-zero M is non-empty'. Let Q be maximal among all annihilators of non-zero elements of M. Then Q is prime and so $Q \in Ass(M)$. [R Noetherian].

Proof: Take such a Q. $Q = Ann(m)$ and suppose $r_1 r_2 \in Q$ and $r_2 \notin Q$.
We show $r_1 \in Q$: $r_1 r_2 \in Q \Rightarrow r_1 r_2 m = 0$
$\Rightarrow r_1 \in Ann(r_2 m)$

However, as $r_2 \notin Q \Rightarrow r_2 m \neq 0$.

But $Q \subseteq Ann(r_2 m)$
Hence Q and $r_1$ lie in $Ann(r_2 m)$
Maximality of $Q \Rightarrow r_1 \in Q$

(former Tripos question!)

(2.20) Lemma: For a non-zero f.g. R-module M with R Noetherian, there is a chain of submodules:

$$0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_s = M$$

with each factor $M_j/M_{j-1} \cong R/P_j$ for some prime $P_j$, not necessarily distinct.

Proof: By (2.19) there is $m_1 \in M$ ($m_1 \neq 0$) with $Ann(m_1) = P_1$ for some prime $P_1$.
Set $M_1 = Rm_1$ and thus $M_1 \cong R/P_1$

Repeat for $M/M_1$ to find $M_2$ with $M_2/M_1 \cong R/P_2$

M is Noetherian, and so the process terminates.

Key fact was using Lemma 2.19.

(2.21) Lemma: If $N \subseteq M$
$\mathrm{Ass}(M) \subseteq \mathrm{Ass}(N) \cup \mathrm{Ass}(M/N)$

Proof: Suppose $P = \mathrm{Ann}(m)$ for some $m \in M$, $P$ prime. Let $M_1 = Rm \cong R/p$. For any $m_1 \in M_1$ $(m_1 \neq 0)$, then $\mathrm{Ann}(m_1) = P$.

If $M_1 \cap N \neq 0$, there exists $m_1 \in M_1 \cap N$, $m_1 \neq 0$ with $\mathrm{Ann}(m_1) = P$. So $P \in \mathrm{Ass}(N)$

If $M_1 \cap N = 0$: then the image of $M_1$ in $M/N$ is isomorphic to $M_1 \cong R/p$ and $\mathrm{Ann}(m + N) = P$. Thus $P \in \mathrm{Ass}(M/N)$.

(2.22) Lemma: $\mathrm{Ass}(M)$ is finite
($M$ is non-zero f.g. $R$-module, $R$ Noetherian).

Proof: Use (2.21) inductively on the chain obtained in (2.20): $\mathrm{Ass}(M) \subseteq \{P_1, \dots, P_s\}$
for the $P_i$ in (2.20).

(2.23) Lemma: $\{\text{minimal primes over } I\} \subseteq \mathrm{Ass}(R/I)$

Proof: By (2.15), there is a product of minimal primes over $I$, (perhaps with repetition), contained in $I$.

$P_1^{s_1} \cdots P_n^{s_n} \subseteq I$ with $P_i \subsetneq P_j$ for $i \neq j$.

Now, consider $\text{Ann}\left(P_2^{s_2} \cdots P_n^{s_n} + I / I\right) = J$

Certainly, $J \geq P_1^{s_1}$. Also, $J P_2^{s_2} \cdots P_n^{s_n} \subseteq I \subseteq P_1$

product

and since $P_1$ is a minimal prime over $I$, $P_1 \neq P_2, \ldots, P_n$

Hence, $J \subseteq P_1$.

Let $M = \left(P_2^{s_2} \cdots P_n^{s_n} + I\right) / I$. Now, use (2.20) to give a chain of submodules of $M$.

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = M$$

with each factor $M_j / M_{j-1} \cong R / Q_j$ with $Q_j$ prime, not necessarily distinct

(note $M \neq 0$ since $J \subseteq P_1 \subsetneq R$)

But $P_1^{s_1}$ annihilates $M$ and hence each factor $M_j / M_{j-1}$ and primeness of $Q_j$ ensures $P_1 \subseteq Q_j \ \forall j$.

Not all the $Q_j \supsetneq P_1$ since $\prod Q_j \subseteq J \subseteq P_1$ and some $Q_j \subseteq P_1$, and hence some $Q_j = P_1$.

Pick at least $j$ with $Q_j = P_1$ and thus $\prod_{k<j} Q_k \nsubseteq P_1$

Take $x \in M_j \setminus M_{j-1}$.

<u>If $j = 1$:</u> then $\text{Ann}(x) = Q_1 = P_1$ and so $P_1 \in \text{Ass}(R/I)$

<u>If $j > 1$:</u> Pick $r \in \left(\prod_{k<j} Q_k\right) \setminus P_1$

Note that $r(sx) = 0$ for any $s \in P_1 = Q_j$.

So $s(rx) = 0$ and $P_1 \subseteq \text{Ann}(rx)$. However $rx \notin M_{j-1}$ since $r \notin P_1$ and $\text{Ann}(rx + M_{j-1}) = P_1$.

So $\mathrm{Ann}(rx) \subseteq P_1$.

Thus: $\mathrm{Ann}(rx) = P_1$ (since $P_1$ inside $\mathrm{Ann}$)

We've shown that $P_1 \in \mathrm{Ass}(M) \subseteq \mathrm{Ass}(R/I)$

[Examples this afternoon at 2pm in MR9.]