Probably Purple Partition Patterns Prefer Prohibiting Perfect Powers

Number Theory Seminar, Charles University

Robin Visser
(joint work with Summer Haag, Praneel Samanta, Swati, Holly Swisher, and Stephanie Treneer)

Faculty of Mathematics and Physics
Charles University

1 April 2026

Part I

When everything was so simple...

Partition numbers

Given a positive integer n, let $p(n)$ denote the number of ways of writing n as a sum of positive integers (ignoring order). We say $p(n)$ is the number of **partitions** of n.

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E.g. We can write **4** as

$$
\begin{aligned}
4 &= 1+1+1+1,\\
4 &= 1+1+2,\\
4 &= 1+3,\\
4 &= 2+2,\\
4 &= 4,
\end{aligned}
$$

thus $p(4) = 5$.

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thus $p(4) = 5$.

E.g. We can write **5** as

$$\begin{aligned} 5 &= 1+1+1+1+1, \\ 5 &= 1+1+1+2, \\ 5 &= 1+1+3, \\ 5 &= 1+2+2, \\ 5 &= 1+4, \\ 5 &= 2+3, \\ 5 &= 5, \end{aligned}$$

thus $p(5) = 7$.

Partition numbers

The first 50 partition numbers:

n	$p(n)$	n	$p(n)$	n	$p(n)$	n	$p(n)$	n	$p(n)$
1	1	11	56	21	792	31	6842	41	44583
2	2	12	77	22	1002	32	8349	42	53174
3	3	13	101	23	1255	33	10143	43	63261
4	5	14	135	24	1575	34	12310	44	75175
5	7	15	176	25	1958	35	14883	45	89134
6	11	16	231	26	2436	36	17977	46	105558
7	15	17	297	27	3010	37	21637	47	124754
8	22	18	395	28	3718	38	26015	48	147273
9	30	19	490	29	4565	39	31185	49	173525
10	42	20	627	30	5604	40	37338	50	204226

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$p(1000) = 24061467864032622473692149727991.$

Questions we may (or may not) care about

- Is there a simple closed-form expression for $p(n)$?
- Is there a recursive formula for $p(n)$?
- Is there an efficient algorithm to compute $p(n)$?
- How fast does $p(n)$ grow as $n \to \infty$?
- Does $p(n)$ exhibit any “random” behaviour?
- Can one predict the multiplicative behaviour (or prime factorisation) of $p(n)$?

Leonhard Euler (1707 – 1783)

Generating function

Note that $p(n)$ is the number of tuples $(k_1, k_2, k_3, \dots)$ of non-negative integers such that

$$n = 1 \cdot k_1 + 2 \cdot k_2 + 3 \cdot k_3 + 4 \cdot k_4 + \dots .$$

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Euler's generating function for $p(n)$:

$$P(x) := \sum_{n=0}^{\infty} p(n)x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + \dots$$

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&= (1 + x + x^2 + x^3 + \dots) \cdot (1 + x^2 + x^4 + x^6 + \dots) \cdot (1 + x^3 + x^6 + x^9 + \dots) \cdots \\
&= \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \\
&= \prod_{k=1}^{\infty} \frac{1}{1-x^k}.
\end{aligned}$$

Recursive formulae for $p(n)$

- Euler's pentagonal theorem:

$$\prod_{k=1}^{\infty}(1-x^k) = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k-1)/2} = 1 + \sum_{k=1}^{\infty}(-1)^k\left(x^{k(3k+1)/2} + x^{k(3k-1)/2}\right)$$
$$= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots$$

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- Using the above and that $P(x)\prod_{k=1}^{\infty}(1-x^k)=1$ we get the recursion:

$$p(n) = \sum_{k\in\mathbb{Z}\setminus\{0\}}(-1)^{k+1}p\left(n-\frac{3k^2+k}{2}\right) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+\cdots.$$

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- By logarithmically differentiating the generating function $P(x)$, we get

$$p(n) = \frac{1}{n}\sum_{k=0}^{n-1}\sigma(n-k)p(k).$$

More fun identities

Let $q(n)$ be the number of ways of writing n as a sum of *distinct* positive integers (ignoring order), and let $Q(x)$ be its generating function .

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$$= \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^6}{1-x^3}\cdot\frac{1-x^8}{1-x^4}\cdot\frac{1-x^{10}}{1-x^5}\cdot\frac{1-x^{12}}{1-x^6}\cdots$$

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$$= \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}\cdot\frac{1}{1-x^9}\cdot\frac{1}{1-x^{11}}\cdots$$

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$$= (1+x+x^2+x^3+\dots)\cdot(1+x^3+x^6+x^9+\dots)\cdot(1+x^5+x^{10}+x^{15}+\dots)\cdots$$

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&= \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}\cdot\frac{1}{1-x^9}\cdot\frac{1}{1-x^{11}}\cdots \\
&= (1+x+x^2+x^3+\dots)\cdot(1+x^3+x^6+x^9+\dots)\cdot(1+x^5+x^{10}+x^{15}+\dots)\cdots \\
&= \sum_{n=0}^{\infty} p_{\text{odd}}(n)x^n.
\end{aligned}
$$

Factorising the partition numbers

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n	$p(n)$	n	$p(n)$	n	$p(n)$	n	$p(n)$
1	1	11	$2^3 \cdot 7$	21	$2^3 \cdot 3^2 \cdot 11$	31	$2 \cdot 11 \cdot 311$
2	2	12	$7 \cdot 11$	22	$2 \cdot 3 \cdot 167$	32	$3 \cdot 11^2 \cdot 23$
3	3	13	101	23	$5 \cdot 251$	33	$3^2 \cdot 7^2 \cdot 23$
4	5	14	$3^3 \cdot 5$	24	$3^2 \cdot 5^2 \cdot 7$	34	$2 \cdot 5 \cdot 1231$
5	7	15	$2^4 \cdot 11$	25	$2 \cdot 11 \cdot 89$	35	$3 \cdot 11^2 \cdot 41$
6	11	16	$3 \cdot 7 \cdot 11$	26	$2^2 \cdot 3 \cdot 7 \cdot 29$	36	17977
7	$3 \cdot 5$	17	$3^3 \cdot 11$	27	$2 \cdot 5 \cdot 7 \cdot 43$	37	$7 \cdot 11 \cdot 281$
8	$2 \cdot 11$	18	$5 \cdot 7 \cdot 11$	28	$2 \cdot 11 \cdot 13^2$	38	$5 \cdot 11^2 \cdot 43$
9	$2 \cdot 3 \cdot 5$	19	$2 \cdot 5 \cdot 7^2$	29	$5 \cdot 11 \cdot 83$	39	$3^4 \cdot 5 \cdot 7 \cdot 11$
10	$2 \cdot 3 \cdot 7$	20	$3 \cdot 11 \cdot 19$	30	$2^2 \cdot 3 \cdot 467$	40	$2 \cdot 3 \cdot 7^2 \cdot 127$

Part II

The man who knew infinity

Srinivasa Ramanujan (1887–1920)

Ramanujan's congruences

"I have proved a number of arithmetic properties of $p(n)$... in particular that

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7}.$$

...I have since found another method which enables me to prove all of these properties and a variety of others, of which the most striking is

$$p(11n+6) \equiv 0 \pmod{11}.$$

There are corresponding properties in which the moduli are powers of 5, 7, or 11.... It appears that there are no equally simple properties for any moduli involving primes other than these three."

- Srinivasa Ramanujan, 1919

Ramanujan's congruences

The first two congruences can be proven via the following ingenious identities:

$$\sum_{k=0}^{\infty} p(5k+4)x^k = 5\frac{\prod_{i=1}^{\infty}(1-x^{5i})^5}{\prod_{i=1}^{\infty}(1-x^i)^6},$$

$$\sum_{k=0}^{\infty} p(7k+5)x^k = 7\frac{\prod_{i=1}^{\infty}(1-x^{7i})^3}{\prod_{i=1}^{\infty}(1-x^i)^4} + 49x\frac{\prod_{i=1}^{\infty}(1-x^{7i})^7}{\prod_{i=1}^{\infty}(1-x^i)^8}.$$

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In the 1960s, Atkin, O'Brien, and Swinnerton-Dyer find further congruences, e.g.:

$$\begin{aligned} p(17303k+237) &\equiv 0 \pmod{13}, \\ p(206839k+2623) &\equiv 0 \pmod{17}, \\ p(1977147619k+815655) &\equiv 0 \pmod{19}. \end{aligned}$$

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- Ahlgren–Boylan (2003) show that, for any prime $\ell \geq 5$,

$$\#\{0 \leq n \leq X : p(n) \equiv r \pmod{\ell}\} \gg_{r,\ell} \begin{cases} X & \text{if } r \equiv 0 \pmod{\ell} \\ \sqrt{X}/\log X & \text{if } r \not\equiv 0 \pmod{\ell} \end{cases}$$

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- Ahlgren–Boylan (2005) show that, for any prime $\ell \geq 5$ and $j \geq 1$, then

$$\#\{0 \leq n \leq X : p(n) \equiv r \pmod{\ell^j}\} \gg_{r,\ell,j} \begin{cases} X & \text{if } r \equiv 0 \pmod{\ell^j} \\ \sqrt{X}/\log X & \text{if } r \not\equiv 0 \pmod{\ell^j} \end{cases}$$

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- Cauchy's theorem gives us

$$p(n) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{P(z)}{z^{n+1}} dz$$

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- By carefully calculating the contributions from the roots of unity, Hardy–Ramanujan gave the following asymptotic formula:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

  as $n \to \infty$

Rademacher's formula

By refining Hardy–Ramanujan's circle method, Rademacher gave the exact formula:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \cdot \frac{d}{dn}\left(\frac{1}{\sqrt{n-\frac{1}{24}}}\sinh\left[\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\;\right]\right),$$

where

$$A_k(n) = \sum_{\substack{0\le m<k \\ (m,k)=1}} \exp\Big(\pi i\left(s(m,k) - 2nm/k\right)\Big),$$

and $s(m,k)$ is the sum

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$$s(m,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{mr}{k} - \left\lfloor \frac{mr}{k} \right\rfloor - \frac{1}{2}\right).$$

This allows one to compute $p(n)$ in $O(n^{1/2+o(1)})$ time!

Large prime factors of $p(n)$

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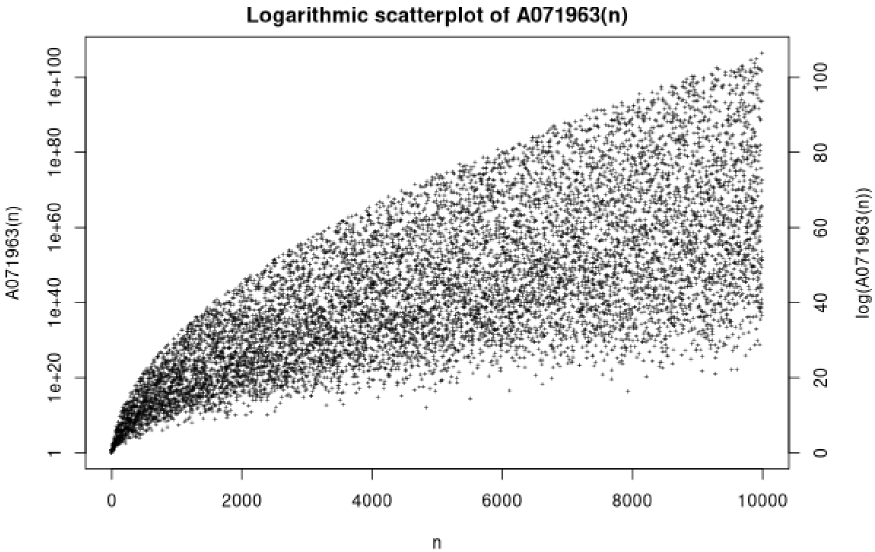
3	3	13	101	23	251	33	23
4	5	14	5	24	7	34	1231
5	7	15	11	25	89	35	41
6	11	16	11	26	29	36	17977
7	5	17	11	27	43	37	281
8	11	18	11	28	13	38	43
9	5	19	7	29	83	39	11
10	7	20	19	30	467	40	127

Large prime factors of $p(n)$



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For almost all n, $P(n) > \log\log\log\log\log\log n$.

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Theorem (Cilleruleo–Luca 2012)

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Part III

The man who loves conjectures

Zhi-Wei Sun (1965-)

A sample of Sun's conjectures

My Favorite Conjecture with $3500 (3500 US dollars) Prize for the First Proof (see OEIS A303389, A303540 and A303821 for similar conjectures)

Any integer $n > 1$ can be written as $a^2 + b^2 + 3^c + 5^d$ with a, b, c, d nonnegative integers. [This has been verified for n up to $2.4*10^{11}$.]

My Four-square Conjecture with $2500 Prize for the First Proof

Every $n = 2, 3, \ldots$ can be written as $x^2 + y^2 + (2^a3^b)^2 + (2^c5^d)^2$, where x, y, a, b, c, d are nonnegative integers. [This has been verified for n up to $1.6*10^{11}$ by Giovanni Resta.]

My 2-4-6-8 Conjecture with $2468 Prize for the First Proof

Any positive integer n can be written as binom(w, 2) + binom(x, 4) + binom(y, 6) + binom(z, 8) with w, x, y, z integers greater than one. [This has been verified for n up to $2*10^{12}$ by Yaakov Baruch.]

My 24-Conjecture with $2400 Prize (see also OEIS A281976 and arXiv:1701.05868)

Any natural number n can be written as the sum of squares of four nonnegative integers x, y, z and w such that both x and x+24y are squares. [This has been verified for n up to 10^{10} by Qing-Hu Hou.]

A sample of Sun's conjectures

My Conjecture involving Primes and Powers of 2 with $1000 Prize (see also Conjecture 3.6(i) of this paper)

Every $n = 2, 3, \ldots$ can be written as a sum of two positive integers k and m such that $2^k + m$ is prime. [This has been verified for n up to 10^7.]

My Conjecture on Alternating Sums of Consecutive Primes with $1000 Prize (see Conj. 1.3 of this published paper)

For any positive integer m, there are consecutive primes $p_k, \ldots, p_n$ $(k < n)$ not exceeding 2m+2.2*sqrt(m) such that $m = p_n - p_{n-1} + \ldots + (-1)^{n-k} p_k$, where p_j denotes the j-th prime. [This has been verified for m up to 10^9 by Chang Zhang.]

My Conjecture on Primitive Roots of the Form x^2+1 with 2000 RMB Prize (see OEIS A239957, A241476 and Conj. 3.1 of this paper)

For any prime p, there is an integer $0 < g < p$ with g-1 an integer square such that g is a primitive root modulo p. [I verified this for all primes below 10^7. Later, C. Greathouse extended the verification to all primes below 10^{10}.]

A sample of Sun's conjectures

My 1680-Conjecture with 1680 RMB Prize (see also OEIS A280831 and Conjecture 4.10(iv) of this published paper)

Any natural number n can be written as the sum of squares of four nonnegative integers x, y, z and w such that $x^4 + 1680y^3z$ is a square. [This has been verified for n up to 10^8 by Qing-Hu Hou.]

My Conjecture on the Representation $n = x^4 + y^3 + z^2 + 2^k$ with \$234 Prize (see also Conjecture 6.1(i) of this paper)

Each n = 2, 3, ... can be written as $x^4 + y^3 + z^2 + 2^k$ with x, y, z nonnegative integers and k a positive integer. [This has been verified by Qing-Hu Hou for n up to 10^9.]

My Conjecture on Primes of the Form x^2+ny^2 with \$200 Prize (see also Conjecture 2.21(i) of this paper)

Each n = 2, 3, ... can be written as x+y with x and y positive integers such that x+ny and x^2+ny^2 are both prime.

My Little 1-3-5 Conjecture with \$135 Prize (see this paper for more such conjectures)

Each n = 0, 1, 2, ... can be written as $x(x+1)/2+y(3y+1)/2+z(5z+1)/2$ with x, y, z nonnegative integers. [I have proved the weaker version with x, y, z integers.]

A sample of Sun's conjectures

My Conjecture related to Bertrand's Postulate with $100 Prize (see also Conjecture 2.18 of this paper)

Let n be any positive integer. Then, for some k=0,...n, both n+k and $n+k^2$ are prime. [I have verified this conjecture for n up to 200,000,000.]

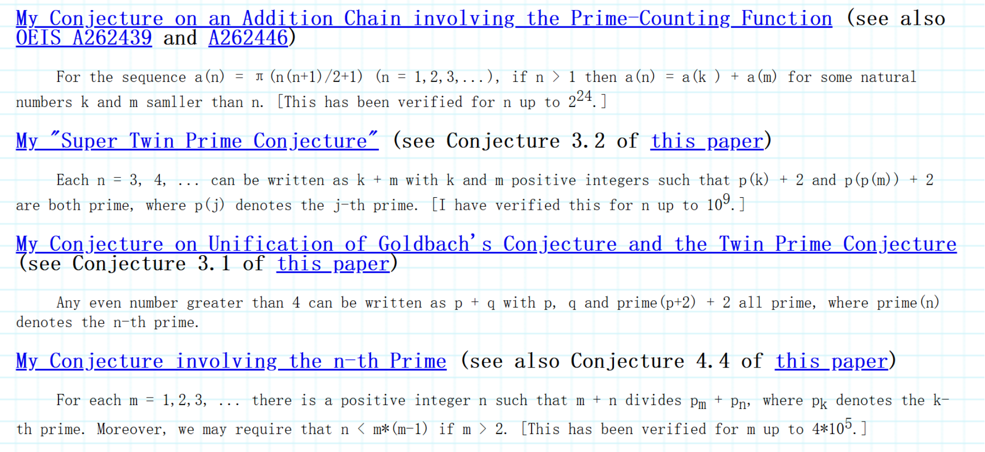
My 100 Conjectures on Representations involving Primes or related Things

My 60 Open Problems on Combinatorial Properties of Primes

My Conjecture on the Prime-Counting Function (see Conjectures 2.1, 2.6 and 2.22 of this paper)

(i) For any integer n>1, $\pi(k*n)$ is prime for some k = 1,...,n, where $\pi(x)$ denotes the number of primes not exceeding x. [I have verified this for n up to 10^7. See OEIS A237578.]
(ii) For every positive integer n, $\pi(\pi(k*n))$ is a square for some k = 1,...,n. [I have verified this for n up to $2*10^5$. See OEIS A238902 and OEIS A239884.]
(iii) For each integer n>2, $\pi(n-p)$ is a square for some prime p < n. [I have verified this for n up to $5*10^8$. See OEIS A237706 and OEIS A237710.]

A sample of Sun's conjectures



OEIS

Search: **"zhi wei sun" conjecture**

Displaying 1-10 of 1574 results found. page 1 2 3 4 5 6 7 8 9 10 ... 101

Sort: relevance | references | number | modified | created Format: long | short | data

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Displaying 1-10 of 1574 results found. page 1 2 3 4 5 6 7 8 9 10 ... 101

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The sequence of partition numbers (A000041):

A000041 a(n) is the number of partitions of n (the partition numbers). 3986
(Formerly M0663 N0244)

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(list; graph; refs; listen; history; text; internal format)

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Conjecture: Each integer n > 2 different from 6 can be written as a sum of finitely many numbers of the form a(k) + 2 (k > 0) with no summand dividing another. This has been verified for n <= 7140. - Zhi-Wei Sun, May 16 2023

Conjecture: No a(n) has the form x^m with m > 1 and x > 1. - Zhi-Wei Sun, Dec 02 2013

Sun's conjecture

Conjecture (Zhi-Wei Sun 2013)

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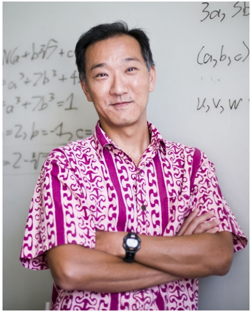
- Ono (2025) showed that the restricted partition function $p_{\mathcal{A}}(n)$ for finite sets $\mathcal{A}$ attains only finitely many perfect powers (under some conditions on k and $\mathcal{A}$).
- Otherwise, pretty much no further progress on this... (I don't even know if there's a positive proportion of $p(n)$ which are *not* perfect powers).

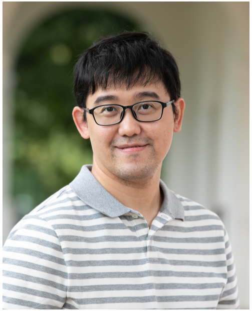
Introducing Merca, Ono, and Tsai

Let's now skip ahead to 2025...

Mircea Merca



Ken Ono



Wei-Lun Tsai

Merca–Ono–Tsai

For a fixed positive integer $k \geq 2$, define $\Delta_k(n)$ as the distance between $p(n)$ and its nearest k-th power, i.e.

$$\Delta_k(n) := \min\left\{|p(n) - m^k| : m \in \mathbb{Z}\right\}.$$

Merca–Ono–Tsai

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$$\Delta_k(n) := \min\left\{|p(n) - m^k| : m \in \mathbb{Z}\right\}.$$

Also define $M_k(d)$ as the largest n such that $p(n)$ is within d to a k-th power, i.e.

$$M_k(d) := \max\left\{n : \Delta_k(n) \leq d\right\}.$$

Some conjectured values of $M_k(d)$

d	$M_2(d)$	$M_3(d)$	$M_4(d)$	$M_5(d)$	$M_6(d)$	$M_7(d)$
0	1	1	1	1	1	1
10^0	35	5	7	2	2	2
10^5	201	133	87	82	64	71
10^{10}	527	295	265	247	227	258
10^{15}	1100	705	482	454	445	388
10^{20}	2058	1019	806	745	654	653
10^{25}	2595	1525	1203	1052	971	978
10^{30}	3804	2135	1636	1564	1337	1244

Merca–Ono–Tsai's conjectures

Conjectures (Merca–Ono–Tsai 2025)

1. For any fixed integers $k \geq 2$ and $d \geq 0$, there are at most finitely many n for which $\Delta_k(n) \leq d$.
2. For any fixed $k > 1$, $M_k(0) = 1$. For every $\varepsilon > 0$, $M_k(d) = o(d^{\varepsilon})$ as $d \to \infty$.
3. For any fixed $d \geq 0$, there exists some L_d for which $M_k(d) = L_d$ for all sufficiently large k.
4. For any fixed $d \geq 0$, there exists a positive integer N_d such that, for $k \geq N_d$ we have

$$M_k(d) = \max\{n : p(n) - 1 \leq d\}.$$

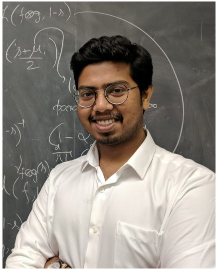
Part IV

and now our attempts...

Holly Swisher

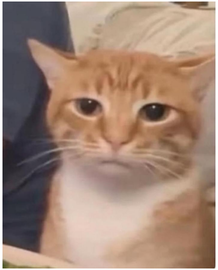
Stephanie Treneer

Summer Haag



Praneel Samanta

Swati



me

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Theorem (Haag–Samanta–Swati–Swisher–Treneer–V 2026)

1. *For any fixed $k \geq 2$, $M_k(d)$ satisfies the lower bound*

$$M_k(d) \geq \frac{3}{2\pi^2}\left(\frac{k}{k-1}\right)^2 (\log d)^2 + O(\log d \log\log d).$$

2. *If $\{M_k(d)\}_{k=1}^{\infty}$ is bounded, there exists a smallest positive integer N_d such that*

$$M_k(d) = \max\{n : p(n) - 1 \leq d\}$$

for all $k \geq N_d$ (i.e. Conjecture 3 implies Conjecture 4).

3. *Furthermore, if N_d exists, then $N_d \geq \lfloor \log_2(d) \rfloor + \left(\frac{1}{\log 2} - o(1)\right) \log\log d$.*

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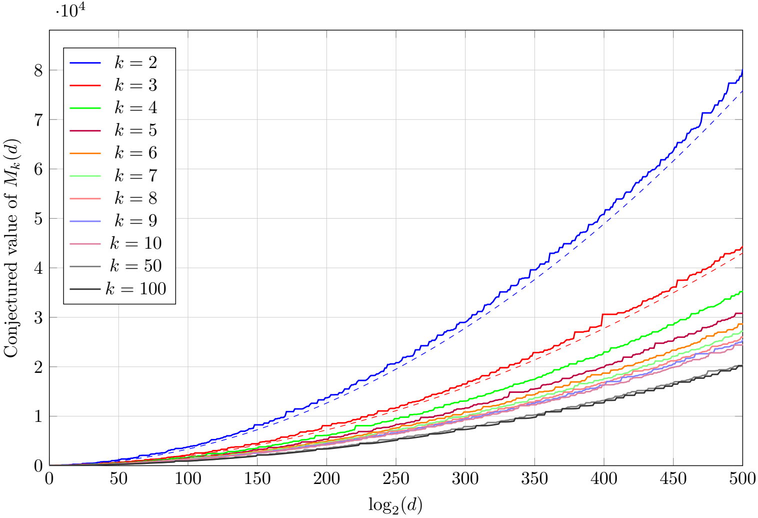
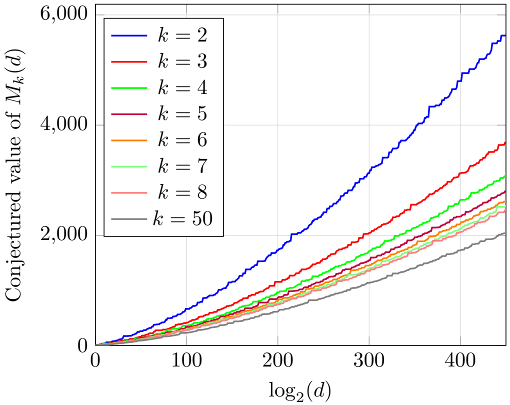
- Analogues of the above theorem hold for any function f such that $f(n) \sim An^{-\beta} \exp(cn^{\gamma})$ as $n \to \infty$, for some constants A, β, c, γ.

Plots of $M_k(d)$

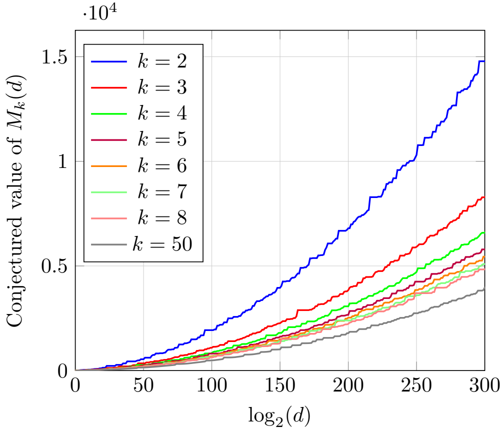


·10^4
Conjectured value of $M_k(d)$
$\log_2(d)$
$k = 2$
$k = 3$
$k = 4$
$k = 5$
$k = 6$
$k = 7$
$k = 8$
$k = 9$
$k = 10$
$k = 50$
$k = 100$
0
1
2
3
4
5
6
7
8
0
50
100
150
200
250
300
350
400
450
500

More plots

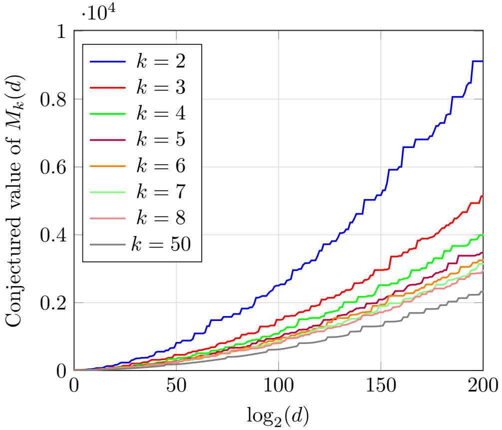


Plane partitions

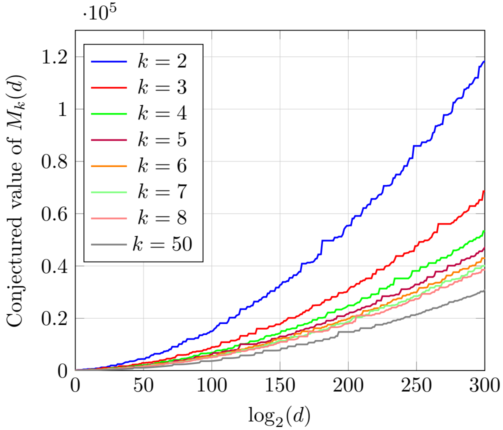


2-coloured partitions

More plots



Overpartitions



Self-conjugate partitions

A heuristic approach

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E.g. some examples of heuristic models:

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- **GUE hypothesis**: Assume that the zeros of the Riemann zeta function $\zeta(s)$ behave like the eigenvalues of a GUE random matrix.
- **Möbius pseudorandomness principle**: Assume that the Möbius function $\mu(n)$ behaves like a random sequence of -1 and $+1$ (for squarefree n).

A simple heuristic model

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- Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ be a sequence of positive real numbers such that $\varepsilon_n \to 0$ as $n \to \infty$. We assume that

$$\varepsilon_n \gg \frac{\log A(n)}{\sqrt{A(n)}} \asymp n \exp\left(-\frac{\pi}{2}\sqrt{\frac{2n}{3}}\right) \quad \text{as } n \to \infty.$$

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- Define a sequence of discrete random variables $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$, where $\mathbf{p}_n$ takes a uniform distribution among the set of integers:

$$\{x \in \mathbb{N} : A(n)(1-\varepsilon_n) \leq x \leq A(n)(1+\varepsilon_n)\}.$$

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- Aim to show that this sequence $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ satisfies the conjectures of Sun and Merca–Ono–Tsai, with probability 1.

Heuristic bounds

Theorem (Haag–Samanta–Swati–Swisher–Treneer–V 2026)

The following statements are each true with probability 1.

(a) *For any $d \geq 0$ and $k \geq 2$, the random variable $\mathbf{M}_k(d)$ is finite.*

(b) *The sequence of random variables $\{\mathbf{p}_n\}_{n=1}^{\infty}$ will take only finitely many perfect powers.*

(c) *Let $k \geq 2$ and assume moreover than $\varepsilon_n \gg A(n)^{-1/k}$ as $n \to \infty$. For any $\varepsilon > 0$ and sufficiently large d, the random variable $\mathbf{M}_k(d)$ satisfies the upper bound*

$$\mathbf{M}_k(d) < (4 + \varepsilon) \cdot \frac{3}{2\pi^2} \left(\frac{k}{k-1} \right)^2 (\log d)^2.$$

In particular, $\mathbf{M}_k(d) \asymp (\log d)^2$ as $d \to \infty$.

(d) *For any fixed $d \geq 0$, the sequence of random variables $\{\mathbf{M}_k(d)\}_{k=2}^{\infty}$ is bounded.*

So... what have we learnt?

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Failing
to solve
Sun's conjecture.

So... what have we learnt?

Further directions

- Obtain a heuristic upper bound for the random variable $\mathbf{N}_d$.
- Refine the heuristic model using the Rademacher expansion for $p(n)$.
- Investigate models where $\mathbf{p}_n$ takes a non-uniform distribution.
- Apply similar heuristic models to other fast-growing functions
- Prove that a positive proportion of $p(n)$ are not perfect powers (or even are not squares).
- Try proving a single value of $M_k(d)$. E.g. $M_2(1) = 35$.

Quotes

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"*Error sending message. An unknown error has occurred.*"

- Claude (2026), when asked for a quote about $p(n)$.

Thank you!