Gaps Between Primes

Warwick Maths Society

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Prime numbers

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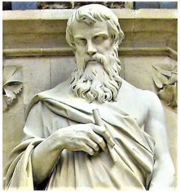
Prime numbers

Theorem (Euclid, $\sim$300BC)

There exist infinitely many prime numbers.



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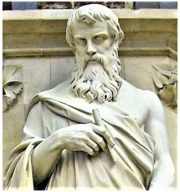

Euclid

Proof: Assume for contradiction there are only finitely many primes $p_1, p_2, \ldots, p_n$. Let $P := p_1 p_2 \cdots p_n$, and let p be a prime factor of $P + 1$. Thus p divides both P and $P + 1$, so p divides 1, contradiction! $\square$

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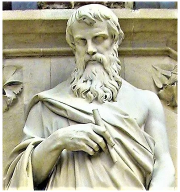

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Nowadays, there are many different proofs by Euler, Erdős, Goldbach, Furstenberg, Pinasco, Whang, Saidak, . . .

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Let $\pi(x)$ denote the number of primes up to x. Then $\pi(x)$ is approximately $x/\log x$.

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- Gauss/Dirichlet conjectured that $\pi(x)$ is approximately $\mathrm{li}(x) = \int_0^x dt/\log t$.

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Limite x	Nombre y par la formule.	Nombre y par les Tables.	Limite x	Nombre y par la formule.	Nombre y par les Tables.
10060	1230	1230	100000	9588	9592
20000	2268	2263	150000	13844	13849
30000	3252	3246	200000	17982	17984
40000	4205	4204	250000	22035	22045
50000	5136	5134	300000	26023	25998
60000	6049	6058	550000	29961	29977
70000	6949	6936	400000	33854	33861
80000	7838	7837			
90000	8717	8713			

Figure: Comparing $\pi(x) + 1$ with $x/(\log x - 1.08366)$ (A.-M. Legendre, *Théorie des Nombres*, 1808)

Prime numbers

Unter	gibt es Primzahlen	Integral $\int \frac{dn}{\log n}$	Abweich.	Ihre Formel	Abweich.
500000	41556	41606,4	+ 50,4	41596,9	+ 40,9
1000000	78501	79627,5	+ 126,5	78672,7	+ 171,7
1500000	114112	114263,1	+ 151,1	114374,0	+ 264,0
2000000	148883	149054,8	+ 171,8	149233,0	+ 350,0
2500000	183016	183245,0	+ 229,0	183495,1	+ 479,1
3000000	216745	216970,6	+ 225,6	217308,5	+ 563,5

Figure: Comparison of $\pi(x)$ with $\int_0^x \frac{dt}{\log t}$ and Dirichlet's conjecture (C. F. Gauss. *Werke*, 1863)

Prime numbers

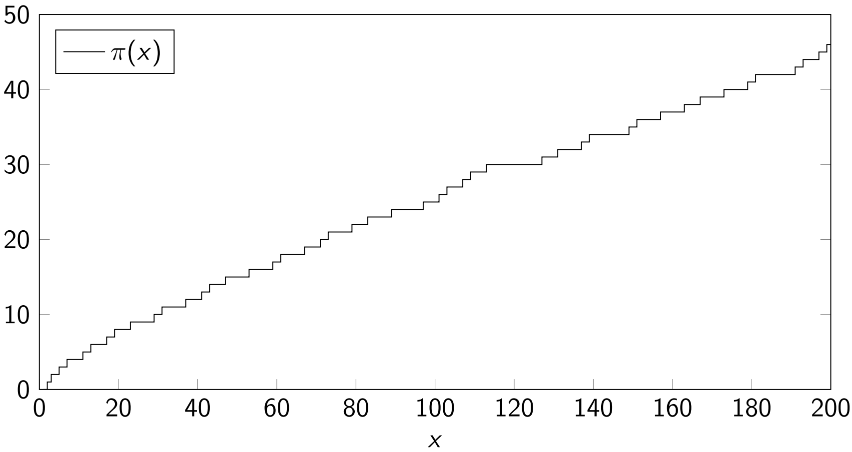

Figure: Comparison of $\pi(x)$ with Legendre/Gauss/Dirichlet's conjectures.

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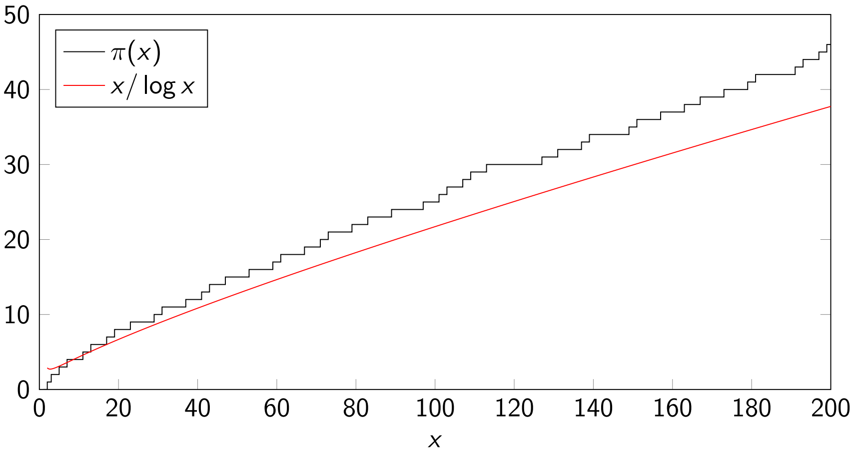

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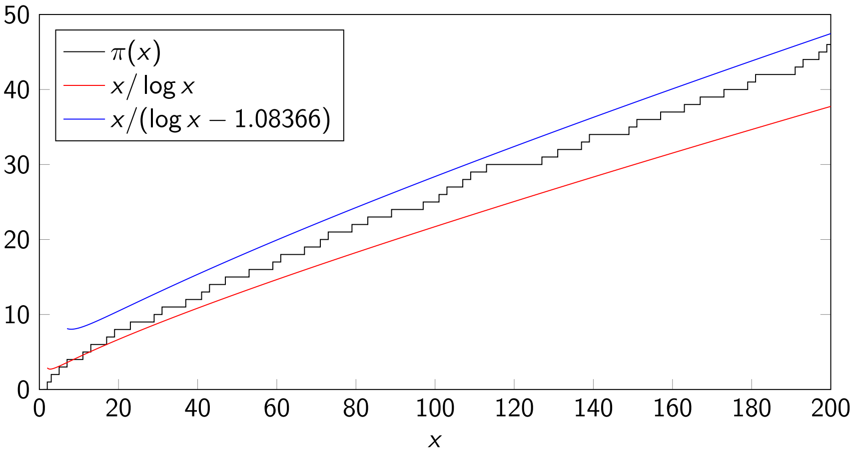

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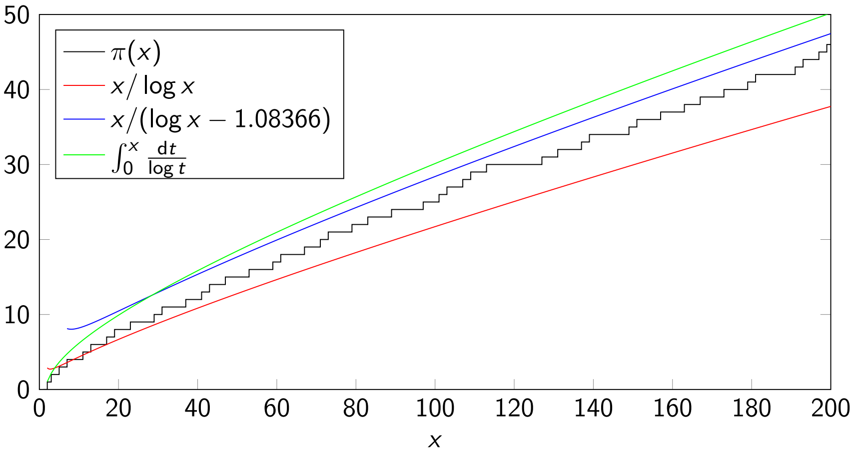

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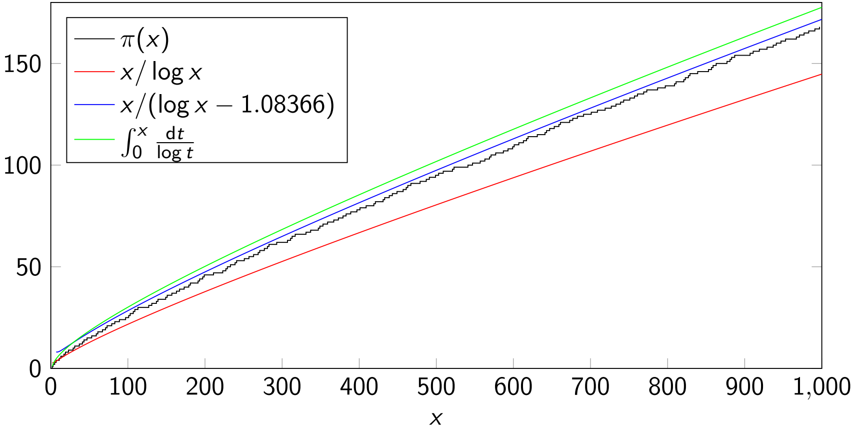

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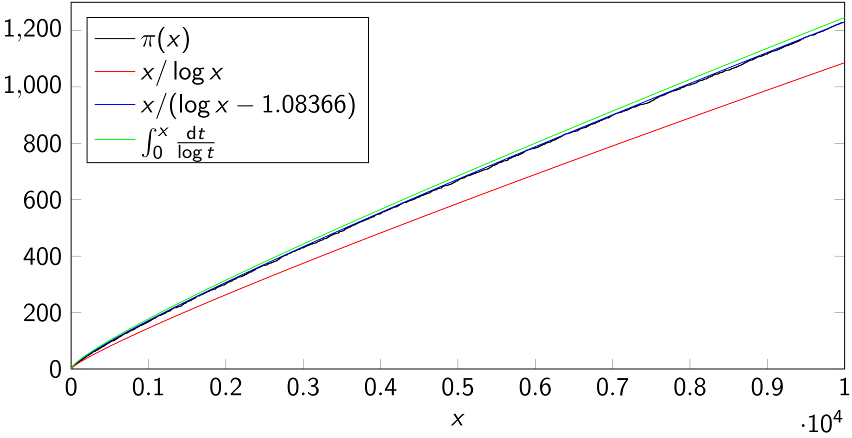

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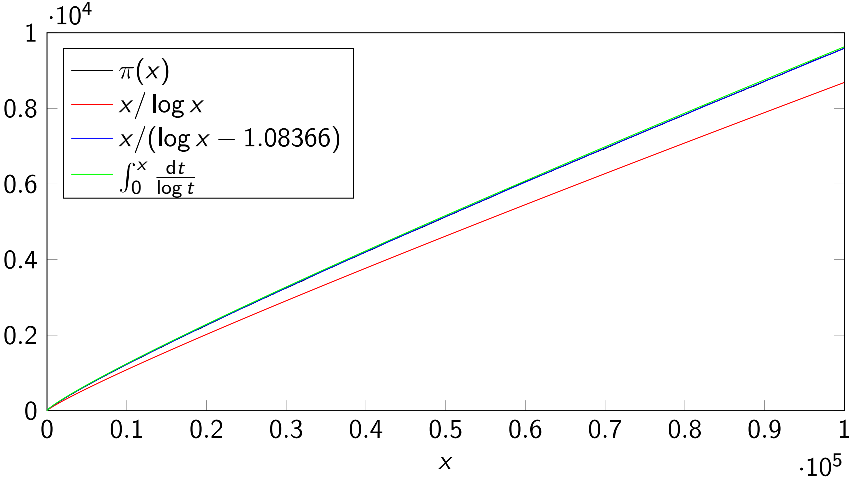

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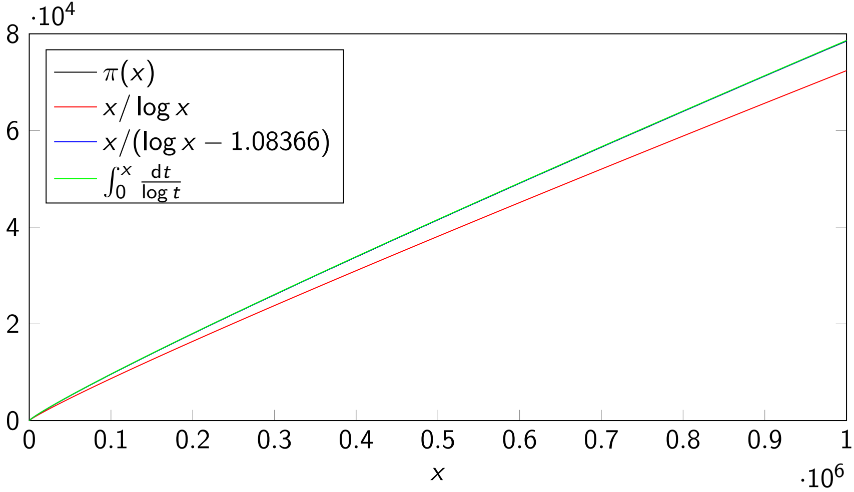

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Conjecture (Bertrand 1845)

For all integers $n > 1$, there exists a prime p between n and $2n$.

Joseph Bertrand

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Theorem (Chebyshev 1852)

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For all sufficiently large x,

$$(0.9212)\frac{x}{\log x} \leq \pi(x) \leq (1.1056)\frac{x}{\log x}.$$

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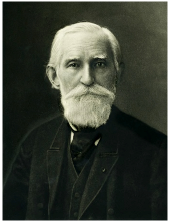

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$$\liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} \leq 1, \quad \textit{and} \quad \limsup_{x\to\infty} \frac{\pi(x)}{x/\log x} \geq 1$$

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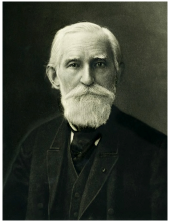

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Prime number theorem

Theorem (Hadamard, de la Vallée Poussin (1896))

Let $\pi(x)$ be the number of primes at most x. Then

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

(equivalently, $p_n \sim n \log n$)

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There are several elementary proofs of the prime number theorem due to Selberg, Erdős (1949) and Newman (1980).

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- How small can g_n be? How goes $g(x) = \min_{p_n \geq x}(p_{n+1} - p_n)$ behave as $x \to \infty$?
- How large can g_n be? How goes $G(x) = \max_{p_n \leq x}(p_{n+1} - p_n)$ behave as $x \to \infty$?

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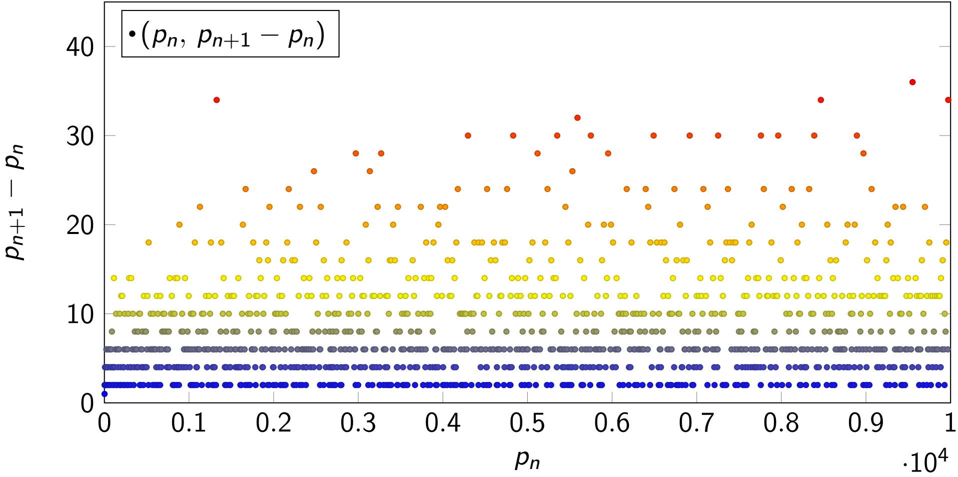

Figure: Scatter plot of $(p_n, p_{n+1} - p_n)$ for all $p_n \leq 10\,000$.

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$$\mathbb{P}(\mathbf{X}_2 = 1) = 1, \quad \text{and} \quad \mathbb{P}(\mathbf{X}_n = 1) = \frac{1}{\log n} \text{ for all } n \geq 3.$$

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Conjecture (Naive Cramér random model)

The asymptotic behaviour for the primes $\{2, 3, 5, 7, \ldots,\}$ should (almost surely) behave like the asymptotic behaviour for the random set $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \ldots\}$.

Cramér's random model

- Assuming Cramér's random model, for a random integer n, we have

$$\mathbb{P}(n \text{ prime}) = \frac{1}{\log n} \quad \text{and} \quad \mathbb{P}(n+2 \text{ prime}) = \frac{1}{\log(n+2)} \sim \frac{1}{\log n}$$

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$$\mathbb{P}(n \text{ and } n+2 \text{ prime}) \sim \frac{1}{(\log n)^2}$$

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- But this is wrong! In particular $\mathbb{P}(n \text{ prime})$ and $\mathbb{P}(n+2 \text{ prime})$ should not be independent events!

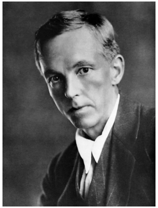
Cramer's random model (modified)

- A more refined model yields the following conjecture:

Conjecture (Hardy–Littlewood)

Let $\pi_2(x)$ denote the number of primes $p \leq x$ such that $p+2$ is prime. Then

$$\pi_2(x) \sim 2 \prod_{\substack{p \text{ prime} \\ p \geq 3}} \left(1 - \frac{1}{(p-1)^2}\right) \frac{x}{(\log x)^2}.$$



G. H. Hardy

John E. Littlewood

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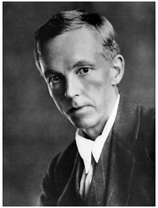
- This conjecture is strongly supported by numerical evidence!



G. H. Hardy

John E. Littlewood

Cramer's random model (modified)

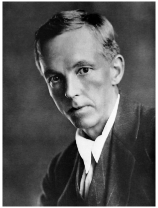
- A more refined model yields the following conjecture:

Conjecture (Hardy–Littlewood)

Let $\pi_2(x)$ denote the number of primes $p \leq x$ such that $p+2$ is prime. Then

$$\pi_2(x) \sim 2 \prod_{\substack{p \text{ prime} \\ p \geq 3}} \left(1 - \frac{1}{(p-1)^2}\right) \frac{x}{(\log x)^2}.$$

- This conjecture is strongly supported by numerical evidence!
- It's known that $\pi_2(x) \leq C \frac{x}{(\log x)^2}$ for some constant $C < 3.4$.



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Small prime gaps

Recall $g(x) = \min_{p_n \geq x} (p_{n+1} - p_n)$.

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For every even integer k, there exists infinitely many primes p such that $p + k$ is prime.

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- Euclid's proof gives $g(n) \leq n! + 1$.
- Bertrand's postulate gives $g(x) \leq x$ for all $x > 2$.
- By the pigeonhole principle, the prime number theorem gives $g(x) \leq (1 + \epsilon) \log x$ for all sufficiently large x.

Jules de Polignac
(Alphonse's daddy)

Small prime gaps

Upper bound for $g(x)$	Authors	Year
$(1+\epsilon)\log x$	Hadamard, de la Vallee Poussin	1896
$(1-c)\log x$	Erdős	1940
$(\frac{57}{59}+\epsilon)\log x$	Rankin	1947
$(\frac{15}{16}+\epsilon)\log x$	Ricci	1954
$(0.4665+\epsilon)\log x$	Bombieri–Davenport	1965
$(0.4571+\epsilon)\log x$	Pilt'ai	1972
$(0.4542+\epsilon)\log x$	Uchiyama	1975
$(0.4425+\epsilon)\log x$	Huxley	1975
$(0.4393+\epsilon)\log x$	Huxley	1984
$(0.2484+\epsilon)\log x$	Maier	1988

Table: Summary of upper bounds for $g(x)$, where $\epsilon > 0$ is any positive real number.

Small prime gaps

Theorem (Goldston–Pintz–Yıldırım 2005)

$$g(x) = o(\log x), \text{ or equivalently } \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0$$

Daniel Goldston

János Pintz

Cem Yıldırım

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- Pintz proved $g(x) \ll (\log x)^{1/3+\epsilon}$ in 2013 (unpublished).

Small prime gaps

Then, a major breakthrough!

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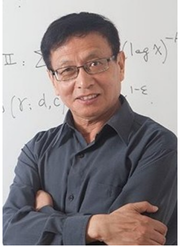
Theorem (Zhang 2013)

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 70\,000\,000.$$



Yitang Zhang

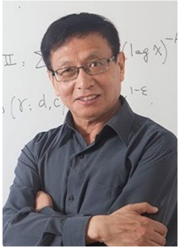
Small prime gaps

Then, a major breakthrough!

Theorem (Zhang 2013)

$$\liminf_{n\to\infty}(p_{n+1}-p_n) \leq 70\,000\,000.$$

- This proves that Polignac's conjecture is true for some even $k \leq 70\,000\,000$!



Yitang Zhang

Small prime gaps

Table: Summary of upper bounds for $\liminf_{n\to\infty}(p_{n+1} - p_n)$

Bound	**Authors**	**Date/Time**
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Small prime gaps

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⋮	⋮	⋮

Polymath

This was organised into a Polymath project, with dozens of contributors!

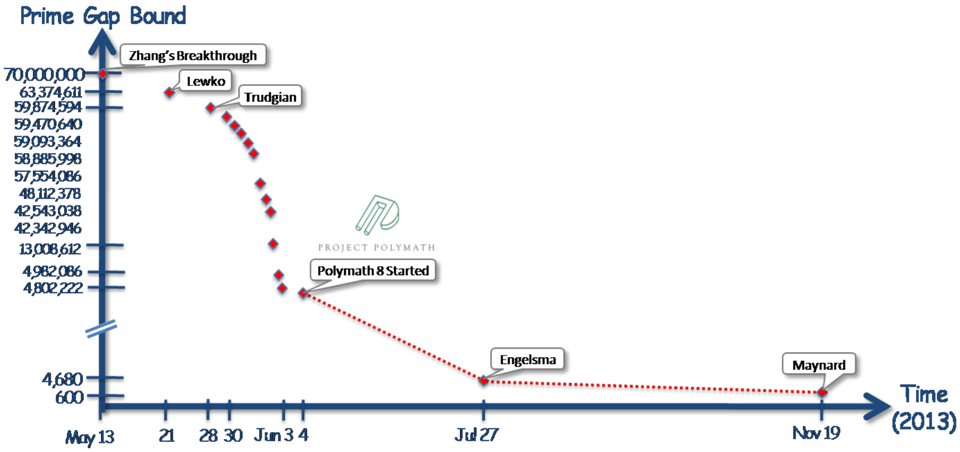

Figure: Progress on upper bound for $\liminf_{n\to\infty}(p_{n+1} - p_n)$ over time.

Small Prime Gaps

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There are infinitely many positive integers n such that $p_{n+1} - p_n \leq 4680$.

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Small Prime Gaps

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- This is the best unconditional bound proven to date!

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- This is the best unconditional bound proven to date!
- Assuming the Elliott-Halberstam conjecture, we have $p_{n+1} - p_n \leq 12$ infinitely often.

Cramer's random model (revisited)

- Recall Cramer's random model, where $\mathbb{P}(\mathbf{X}_n = 1) = 1/\log n$ and the random variables $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \ldots$ are defined as

$$\mathbf{P}_1 = 2, \quad \text{and} \quad \mathbf{P}_{n+1} = \min\{i : \mathbf{X}_i = 1 \text{ and } i > \mathbf{P}_n\}$$

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The following holds with probability 1:

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Theorem (Cramer 1936)

The following holds with probability 1:

$$\limsup_{x \to \infty} \frac{\mathbf{G}(x)}{(\log x)^2} = 1.$$

- However, we've seen Cramér's model isn't perfect!

Granville's random model

- Granville proposed the following refinement: For a suitably chosen parameter T, let $\mathbf{X}_3, \mathbf{X}_4, \ldots$ be a sequence of random variables such that, if n has some prime factor $\leq T$, then $\mathbf{X}_n = 0$, otherwise, let

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"*It is evident that the primes are randomly distributed but, unfortunately, we don't know what 'random' means.*" - R. C. Vaughan (February 1990)

Results

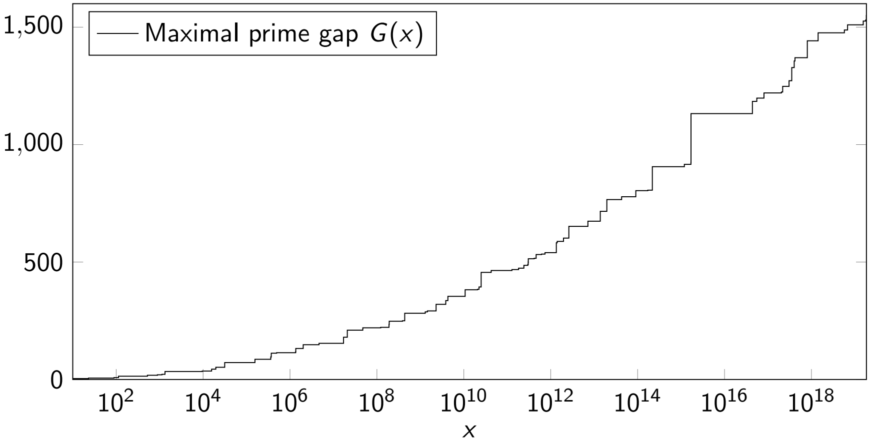

Figure: Comparison of $G(x)$ with Cramer and Granville's conjecture.

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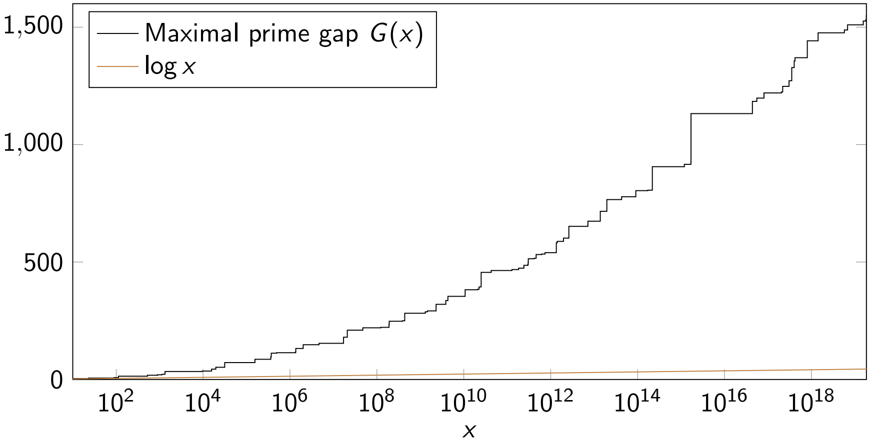

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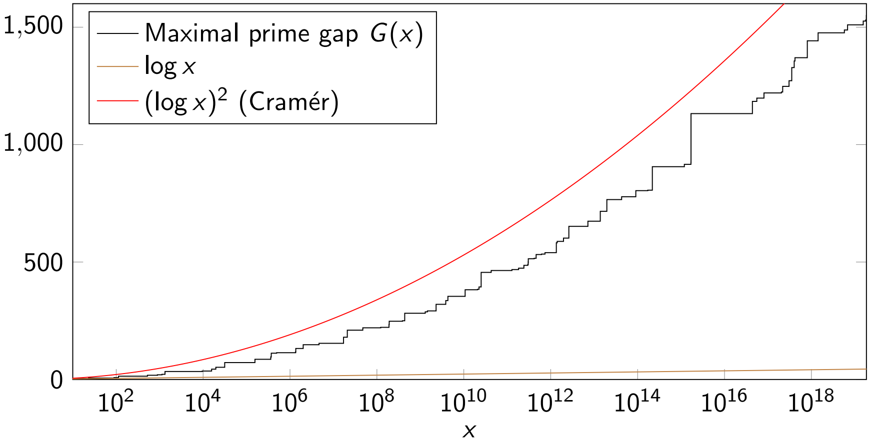

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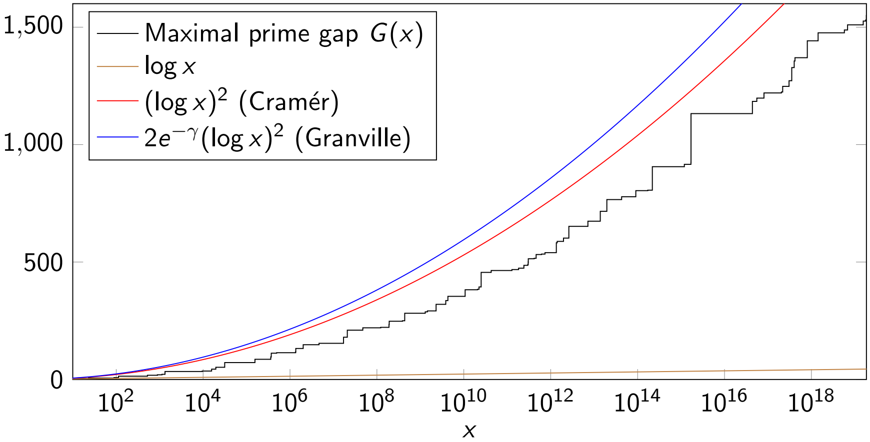


Figure: Comparison of $G(x)$ with Cramer and Granville's conjecture.

Large prime gaps

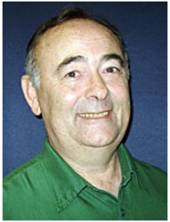
Table: Summary of upper bounds of the form $G(x) \ll x^{\theta}$ proven to date.

Constant θ		Authors	Year
$1 - 1/33000$	$\approx 0.999969\ldots$	Hoheisel	1930
$1 - 1/250$	$= 0.996$	Heilbronn	1933
$3/4 + \epsilon$	$= 0.75$	Chudukov	1936
$5/8 + \epsilon$	$= 0.625$	Ingham	1937
$5/8 - 1/616 + \epsilon$	$\approx 0.623377\ldots$	Titchmarsh	1942
$5/8 - 1/488 + \epsilon$	$\approx 0.622951\ldots$	Min	1949
$5/8 - 1/392 + \epsilon$	$\approx 0.622449\ldots$	Haneke	1962
$3/5 + \epsilon$	$= 0.6$	Montgomery	1971
$7/12 + \epsilon$	$\approx 0.583333\ldots$	Huxley	1972
$13/23$	$\approx 0.565217\ldots$	Iwaniec, Jutila	1979
$11/20$	$= 0.55$	Heath-Brown, Iwaniec	1979
$11/20 - 1/406$	$\approx 0.547537\ldots$	Iwaniec, Pintz	1984
$11/20 - 1/384$	$\approx 0.547396\ldots$	Mozzochi	1986
$6/11$	$\approx 0.545454\ldots$	Lou, Yao	1992
$107/200$	$= 0.535$	Baker, Harman	1996

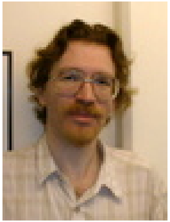
Large prime gaps

Theorem (Baker–Harman–Pintz (2001))

$G(x) \ll x^{0.525}$



Roger Baker



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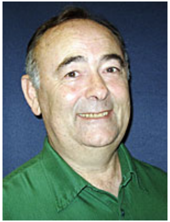

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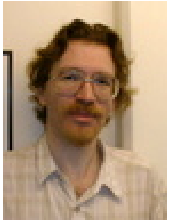

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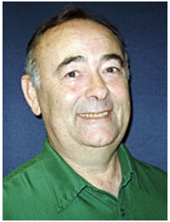

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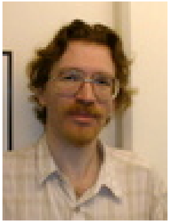

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János Pintz

- Assuming the Riemann Hypothessis, we get $G(x) \ll \sqrt{x}\log x$.
- Assuming both RH and some results on Montgomery's pair correlation function, we get $G(x) \ll \sqrt{x \log x}$.

Large prime gaps

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For any positive integer n, there exists n consecutive composite numbers (i.e. $G(x) \to \infty$ as $x \to \infty$).

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- Chebyshev proved $G(x) \gg \log x$.
- The prime number theorem implies $G(x) \geq (1-\epsilon)\log x$

Large prime gaps

Definition

For any positive integer x, let $Y(x)$ be the largest integer y such that the interval $\{1, 2, \ldots, y\}$ can be sieved out by a set of residue classes $a_p \bmod p$ for each prime $p \leq x$.

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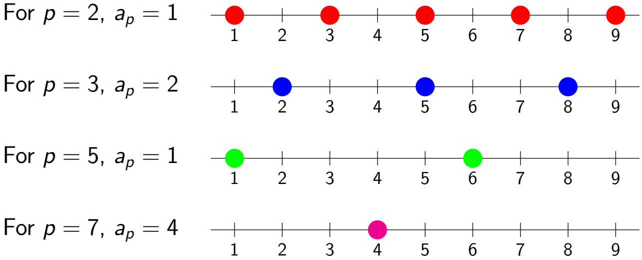
Example: Let $x = 7$. The primes $\leq x$ are $\{2, 3, 5, 7\}$. We can cover the interval $\{1, \ldots, 9\}$ by choosing the following residue classes: $a_2 = 1$, $a_3 = 2$, $a_5 = 1$, and $a_7 = 4$



Large prime gaps

Definition

For any positive integer x, let $Y(x)$ be the largest integer y such that the interval $\{1, 2, \ldots, y\}$ can be sieved out by a set of residue classes $a_p \bmod p$ for each prime $p \leq x$.

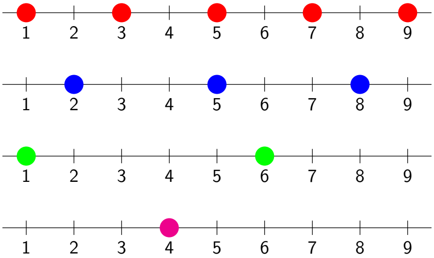
Example: Let $x = 7$. The primes $\leq x$ are $\{2, 3, 5, 7\}$. We can cover the interval $\{1, \ldots, 9\}$ by choosing the following residue classes: $a_2 = 1$, $a_3 = 2$, $a_5 = 1$, and $a_7 = 4$

For $p = 2$, $a_p = 1$

For $p = 3$, $a_p = 2$

For $p = 5$, $a_p = 1$

For $p = 7$, $a_p = 4$



As no choice of residue classes can cover $\{1, \ldots, 10\}$, this proves $Y(7) = 9$.

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Theorem

Let $\epsilon > 0$*. Then* $G(x) \geq Y((1-\epsilon)\log x)$ *for all sufficiently large* x*.*

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- Let $p_1, p_2, \ldots, p_n$ be all the primes $\leq x$, and let $P = p_1 p_2 \cdots p_n$.
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- We claim all the integers $\{m+1, m+2, \ldots, m+Y(x)\}$ are composite. Indeed, for any $i = 1, \ldots, Y(x)$, there exists a $p \leq x$ such that p divides $m+i$.

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- This gives a sequence of $Y(x)$ consecutive composite numbers, no larger than P, and thus $G(P) \geq Y(x)$.
- By the prime number theorem, $\log P \sim x$, which proves $G(x) \geq Y((1-\epsilon)\log x)$. □

Large prime gaps

Table: Summary of lower bounds for $G(x)$ arising from lower bounds for $Y(x)$.

Bound for $G(x)$	**Bound for $Y(x)$**	**Authors**	**Year**
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$\frac{\log x \log\log x}{(\log\log\log x)^2}$	$\frac{x \log x}{(\log\log x)^2}$	Erdős (Chang)	1935 (1938)

Large prime gaps

Theorem (Rankin 1938)

For any $\epsilon > 0$,

$$G(x) \geq (c - \epsilon)\frac{\log x \log\log x \log\log\log\log x}{(\log\log\log x)^2}$$

where $c = \frac{1}{3}$, *for sufficiently large* x.

R.A. Rankin

Large prime gaps

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Table: Summary of improvements made to the constant c.

Constant c		**Authors**	**Year**
$\frac{1}{2}e^{\gamma}$	≈ 0.8905	Schonhage	1963
e^{γ}	≈ 1.7811	Rankin	1963
$1.31256e^{\gamma}$	≈ 2.0172	Maier, Pomerance	1990
$2e^{\gamma}$	≈ 3.5621	Pintz	1997

Large prime gaps

Erdős offered a cash prize of 10 000 USD for anyone who could prove c can be arbitrary large!

Rankin [35] proved that for some $c > 0$ and infinitely many n the following inequality holds:

$$p_{n+1} - p_n > \frac{c \log n \log\log n \log\log\log\log n}{(\log\log\log n)^2}. \tag{1}$$

I offered (perhaps somewhat rashly) \$10 000 for a proof that (1) holds for every c. The

- Excerpt from *A Tribute to Paul Erdős* (1990)

Large prime gaps

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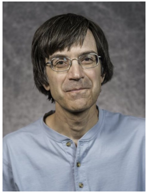
Theorem (Ford–Green–Konyagin–Tao, Maynard 2014)

$$G(x) \geq f(x) \frac{\log x \log \log x \log \log \log \log x}{(\log \log \log x)^2}$$

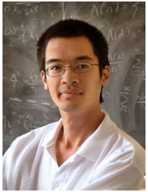
for some function $f(x)$ that goes to infinity as $x \to \infty$.



Kevin Ford

Ben Green

Sergei Konyagin



Terence Tao

James Maynard

Large prime gaps

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Terence Tao has offered 10 000 USD to anyone who can prove the implicit constant given above can be arbitrarily large!

Conclusion

So far we have $2 \le g(x) \le 246$ and

$$\frac{\log x \log\log x \log\log\log\log x}{\log\log\log x} \ll G(x) \ll x^{0.525}.$$

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Thank you!