Sums of three cubes

Warwick Maths Society

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15 November 2022

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What about $k = 4$ or $k = 5$? All integer cubes are $0, \pm 1 \bmod 9$, so $x^3 + y^3 + z^3 = k$ has no solutions if $k \equiv \pm 4 \bmod 9$.

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Question

Let $k \in \mathbb{Z}$, $k \not\equiv \pm 4 \bmod 9$. Does there exist $x, y, z \in \mathbb{Z}$ such that $x^3 + y^3 + z^3 = k$?

Sum of five cubes

Theorem

Let $k \in \mathbb{Z}$. Then there exist infinitely many $v, w, x, y, z \in \mathbb{Z}$ such that $v^3 + w^3 + x^3 + y^3 + z^3 = k$.

Sum of five cubes

Theorem

Let $k \in \mathbb{Z}$. Then there exist infinitely many $v, w, x, y, z \in \mathbb{Z}$ such that $v^3 + w^3 + x^3 + y^3 + z^3 = k$.

Proof: Every multiple of 6 is the sum of four cubes:

$$6n = (n+1)^3 + (n-1)^3 - n^3 - n^3$$

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Note that $k \equiv k^3 \pmod 6$. Therefore, for any integer $r \in \mathbb{Z}$

$$k = (k+6r)^3 + (n+1)^3 + (n-1)^3 - n^3 - n^3$$

where $n := \frac{k-(k+6r)^3}{6} \in \mathbb{Z}$. □

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- Moreover, this proves that every $k \in \mathbb{Z}$ has the form $k = 2w^3 + x^3 + y^3 + z^3$.

Sum of four cubes

Theorem (Demjanenko (1966))

Let $k \in \mathbb{Z}$, $k \not\equiv \pm 4$ *mod 9. Then there exist infinitely many* $w, x, y, z \in \mathbb{Z}$ *such that* $w^3 + x^3 + y^3 + z^3 = k$.

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Proof (sketch): Can give explicit identities for expressing

$$6n,\ 6n+3,\ 18n+1,\ 18n+7,\ 18n+8,\ 54n+2,\ 54n+20,\ 216n-16,\ 216n+92$$

as sums of four cubes.

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Conjecture

Let $k \in \mathbb{Z}$.

1. (*Weaker version*) There exist $w, x, y, z \in \mathbb{Z}$ such that $w^3 + x^3 + y^3 + z^3 = k$.

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Let $k \in \mathbb{Z}$.

1. (*Weaker version*) There exist $w, x, y, z \in \mathbb{Z}$ such that $w^3 + x^3 + y^3 + z^3 = k$.
2. (*Stronger version*) There exist $x, y, z \in \mathbb{Z}$ such that $2x^3 + y^3 + z^3 = k$.

Sum of two cubes

Theorem

Let $k \in \mathbb{Z}$. Then there exist $x, y \in \mathbb{Z}$ such that $x^3 + y^3 = k$ if and only if there exists a divisor $d|k$ such that $\frac{4k/d - d^2}{3}$ is an integral square.

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Proof: Note that $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$. So k is sum of two integer cubes if and only if there exists $x, y \in \mathbb{Z}$ such that

$$x + y = d,$$
$$x^2 - xy + y^2 = k/d,$$

for some divisor d of k.

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$$x + y = d,$$
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for some divisor d of k. Solving for x yields

$$x = \frac{d}{2} \pm \frac{1}{2}\sqrt{\frac{4k/d - d^2}{3}}.$$

□

Sum of three rational cubes

One of the earliest results on sums of three cubes is finding rational solutions. The following is a theorem of S. Ryley (a schoolmaster of Leeds):

Theorem (Ryley (1825))

Every rational number $k \in \mathbb{Q}$ is the sum of three rational cubes in infinitely many ways.

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Proof: For any $n \in \mathbb{Q}^\times$, note that

$$k = \left(\frac{27k^3 - n^9}{27k^2n^2 + 9kn^5 + 3n^8}\right)^3 + \left(\frac{-27k^3 + 9kn^6 + n^9}{27k^2n^2 + 9kn^5 + 3n^8}\right)^3 + \left(\frac{27k^2n^3 + 9kn^6}{27k^2n^2 + 9kn^5 + 3n^8}\right)^3.$$

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□

- Can also prove every positive rational is sum of three *positive* rational cubes.
- Not all integers are sums of two rational cubes (e.g. primes which are 2 or 5 mod 9).

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_0	2_0	3_0	4_0	5_0	6_0	7_0	8_0	9_0	10_0	11_0	12_0	13_0	14_0	15_0	16_0	17_0	18_0
19_0	20_0	21_0	22_0	23_0	24_0	25_0	26_0	27_0	28_0	29_0	30_0	31_0	32_0	33_0	34_0	35_0	36_0
37_0	38_0	39_0	40_0	41_0	42_0	43_0	44_0	45_0	46_0	47_0	48_0	49_0	50_0	51_0	52_0	53_0	54_0
55_0	56_0	57_0	58_0	59_0	60_0	61_0	62_0	63_0	64_0	65_0	66_0	67_0	68_0	69_0	70_0	71_0	72_0
73_0	74_0	75_0	76_0	77_0	78_0	79_0	80_0	81_0	82_0	83_0	84_0	85_0	86_0	87_0	88_0	89_0	90_0
91_0	92_0	93_0	94_0	95_0	96_0	97_0	98_0	99_0	100_0	101_0	102_0	103_0	104_0	105_0	106_0	107_0	108_0
109_0	110_0	111_0	112_0	113_0	114_0	115_0	116_0	117_0	118_0	119_0	120_0	121_0	122_0	123_0	124_0	125_0	126_0

(*complete up to* $|x|, |y|, |z| \leq 0$)

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19 $_0$	20 $_0$	21 $_0$	*22*	*23*	24 $_0$	25 $_0$	26 $_0$	27 $_0$	28 $_0$	29 $_0$	30 $_0$	*31*	*32*	33 $_0$	34 $_0$	35 $_0$	36 $_0$
37 $_0$	38 $_0$	39 $_0$	*40*	*41*	42 $_0$	43 $_0$	44 $_0$	45 $_0$	46 $_0$	47 $_0$	48 $_0$	*49*	*50*	51 $_0$	52 $_0$	53 $_0$	54 $_0$
55 $_0$	56 $_0$	57 $_0$	*58*	*59*	60 $_0$	61 $_0$	62 $_0$	63 $_0$	64 $_0$	65 $_0$	66 $_0$	*67*	*68*	69 $_0$	70 $_0$	71 $_0$	72 $_0$
73 $_0$	74 $_0$	75 $_0$	*76*	*77*	78 $_0$	79 $_0$	80 $_0$	81 $_0$	82 $_0$	83 $_0$	84 $_0$	*85*	*86*	87 $_0$	88 $_0$	89 $_0$	90 $_0$
91 $_0$	92 $_0$	93 $_0$	*94*	*95*	96 $_0$	97 $_0$	98 $_0$	99 $_0$	100 $_0$	101 $_0$	102 $_0$	*103*	*104*	105 $_0$	106 $_0$	107 $_0$	108 $_0$
109 $_0$	110 $_0$	111 $_0$	*112*	*113*	114 $_0$	115 $_0$	116 $_0$	117 $_0$	118 $_0$	119 $_0$	120 $_0$	*121*	*122*	123 $_0$	124 $_0$	125 $_0$	126 $_0$

(*complete up to* $|x|, |y|, |z| \leq 0$)

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Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{2}	2_{1}	3_{1}	*4*	*5*	6_{0}	7_{0}	8_{0}	9_{0}	10_{0}	11_{0}	12_{0}	*13*	*14*	15_{0}	16_{0}	17_{0}	18_{0}
19_{0}	20_{0}	21_{0}	*22*	*23*	24_{0}	25_{0}	26_{0}	27_{0}	28_{0}	29_{0}	30_{0}	*31*	*32*	33_{0}	34_{0}	35_{0}	36_{0}
37_{0}	38_{0}	39_{0}	*40*	*41*	42_{0}	43_{0}	44_{0}	45_{0}	46_{0}	47_{0}	48_{0}	*49*	*50*	51_{0}	52_{0}	53_{0}	54_{0}
55_{0}	56_{0}	57_{0}	*58*	*59*	60_{0}	61_{0}	62_{0}	63_{0}	64_{0}	65_{0}	66_{0}	*67*	*68*	69_{0}	70_{0}	71_{0}	72_{0}
73_{0}	74_{0}	75_{0}	*76*	*77*	78_{0}	79_{0}	80_{0}	81_{0}	82_{0}	83_{0}	84_{0}	*85*	*86*	87_{0}	88_{0}	89_{0}	90_{0}
91_{0}	92_{0}	93_{0}	*94*	*95*	96_{0}	97_{0}	98_{0}	99_{0}	100_{0}	101_{0}	102_{0}	*103*	*104*	105_{0}	106_{0}	107_{0}	108_{0}
109_{0}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{0}	116_{0}	117_{0}	118_{0}	119_{0}	120_{0}	*121*	*122*	123_{0}	124_{0}	125_{0}	126_{0}

(*complete up to* $|x|, |y|, |z| \leq 1$)

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19_{0}	20_{0}	21_{0}	*22*	*23*	24_{1}	25_{0}	26_{0}	27_{0}	28_{0}	29_{0}	30_{0}	*31*	*32*	33_{0}	34_{0}	35_{0}	36_{0}
37_{0}	38_{0}	39_{0}	*40*	*41*	42_{0}	43_{0}	44_{0}	45_{0}	46_{0}	47_{0}	48_{0}	*49*	*50*	51_{0}	52_{0}	53_{0}	54_{0}
55_{0}	56_{0}	57_{0}	*58*	*59*	60_{0}	61_{0}	62_{0}	63_{0}	64_{0}	65_{0}	66_{0}	*67*	*68*	69_{0}	70_{0}	71_{0}	72_{0}
73_{0}	74_{0}	75_{0}	*76*	*77*	78_{0}	79_{0}	80_{0}	81_{0}	82_{0}	83_{0}	84_{0}	*85*	*86*	87_{0}	88_{0}	89_{0}	90_{0}
91_{0}	92_{0}	93_{0}	*94*	*95*	96_{0}	97_{0}	98_{0}	99_{0}	100_{0}	101_{0}	102_{0}	*103*	*104*	105_{0}	106_{0}	107_{0}	108_{0}
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(*complete up to* $|x|, |y|, |z| \leq 2$)

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19_{1}	20_{1}	21_{0}	*22*	*23*	24_{1}	25_{1}	26_{1}	27_{4}	28_{1}	29_{1}	30_{0}	*31*	*32*	33_{0}	34_{1}	35_{1}	36_{1}
37_{0}	38_{0}	39_{0}	*40*	*41*	42_{0}	43_{1}	44_{0}	45_{0}	46_{1}	47_{0}	48_{0}	*49*	*50*	51_{0}	52_{0}	53_{1}	54_{1}
55_{1}	56_{0}	57_{0}	*58*	*59*	60_{0}	61_{0}	62_{1}	63_{0}	64_{0}	65_{0}	66_{0}	*67*	*68*	69_{0}	70_{0}	71_{0}	72_{0}
73_{0}	74_{0}	75_{0}	*76*	*77*	78_{0}	79_{0}	80_{0}	81_{1}	82_{0}	83_{0}	84_{0}	*85*	*86*	87_{0}	88_{0}	89_{0}	90_{0}
91_{0}	92_{0}	93_{0}	*94*	*95*	96_{0}	97_{0}	98_{0}	99_{0}	100_{0}	101_{0}	102_{0}	*103*	*104*	105_{0}	106_{0}	107_{0}	108_{0}
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19 1	20 1	21 0	*22*	*23*	24 1	25 1	26 1	27 5	28 1	29 2	30 0	*31*	*32*	33 0	34 1	35 1	36 2
37 1	38 1	39 0	*40*	*41*	42 0	43 1	44 0	45 1	46 1	47 0	48 1	*49*	*50*	51 0	52 0	53 1	54 1
55 2	56 1	57 1	*58*	*59*	60 0	61 0	62 2	63 1	64 5	65 1	66 1	*67*	*68*	69 0	70 0	71 1	72 1
73 1	74 0	75 0	*76*	*77*	78 0	79 0	80 1	81 1	82 0	83 1	84 0	*85*	*86*	87 0	88 0	89 0	90 1
91 1	92 1	93 0	*94*	*95*	96 0	97 0	98 0	99 1	100 0	101 1	102 0	*103*	*104*	105 0	106 0	107 0	108 0
109 0	110 0	111 0	*112*	*113*	114 0	115 0	116 0	117 0	118 1	119 0	120 1	*121*	*122*	123 0	124 0	125 0	126 0

(*complete up to* $|x|, |y|, |z| \leq 4$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \le y \le z$.

1_{6}	2_{1}	3_{2}	*4*	*5*	6_{1}	7_{1}	8_{6}	9_{1}	10_{2}	11_{1}	12_{0}	*13*	*14*	15_{1}	16_{1}	17_{1}	18_{1}
19_{1}	20_{1}	21_{0}	*22*	*23*	24_{1}	25_{1}	26_{1}	27_{6}	28_{1}	29_{2}	30_{0}	*31*	*32*	33_{0}	34_{2}	35_{1}	36_{2}
37_{1}	38_{1}	39_{0}	*40*	*41*	42_{0}	43_{1}	44_{0}	45_{1}	46_{1}	47_{0}	48_{1}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{1}
55_{2}	56_{1}	57_{1}	*58*	*59*	60_{1}	61_{1}	62_{3}	63_{1}	64_{6}	65_{1}	66_{1}	*67*	*68*	69_{1}	70_{0}	71_{2}	72_{1}
73_{1}	74_{0}	75_{0}	*76*	*77*	78_{0}	79_{0}	80_{1}	81_{1}	82_{0}	83_{1}	84_{0}	*85*	*86*	87_{0}	88_{1}	89_{0}	90_{2}
91_{1}	92_{1}	93_{0}	*94*	*95*	96_{0}	97_{1}	98_{1}	99_{2}	100_{0}	101_{1}	102_{0}	*103*	*104*	105_{0}	106_{1}	107_{0}	108_{0}
109_{1}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{0}	116_{1}	117_{1}	118_{2}	119_{0}	120_{1}	*121*	*122*	123_{1}	124_{1}	125_{6}	126_{1}

(*complete up to* $|x|, |y|, |z| \le 5$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{7}	2_{1}	3_{2}	*4*	*5*	6_{1}	7_{1}	8_{7}	9_{1}	10_{2}	11_{1}	12_{0}	*13*	*14*	15_{1}	16_{1}	17_{1}	18_{1}
19_{1}	20_{1}	21_{0}	*22*	*23*	24_{1}	25_{1}	26_{1}	27_{8}	28_{1}	29_{2}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{1}	36_{2}
37_{1}	38_{1}	39_{0}	*40*	*41*	42_{0}	43_{1}	44_{0}	45_{1}	46_{1}	47_{0}	48_{1}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{1}
55_{2}	56_{1}	57_{1}	*58*	*59*	60_{1}	61_{1}	62_{3}	63_{1}	64_{8}	65_{1}	66_{1}	*67*	*68*	69_{1}	70_{0}	71_{2}	72_{1}
73_{1}	74_{0}	75_{0}	*76*	*77*	78_{0}	79_{0}	80_{1}	81_{1}	82_{0}	83_{2}	84_{0}	*85*	*86*	87_{0}	88_{2}	89_{0}	90_{3}
91_{2}	92_{2}	93_{0}	*94*	*95*	96_{0}	97_{1}	98_{1}	99_{3}	100_{0}	101_{1}	102_{0}	*103*	*104*	105_{0}	106_{1}	107_{0}	108_{0}
109_{1}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{0}	116_{1}	117_{1}	118_{3}	119_{0}	120_{1}	*121*	*122*	123_{1}	124_{1}	125_{8}	126_{1}

(*complete up to* $|x|, |y|, |z| \leq 6$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{8}	2_{2}	3_{2}	*4*	*5*	6_{1}	7_{1}	8_{8}	9_{1}	10_{2}	11_{1}	12_{0}	*13*	*14*	15_{1}	16_{1}	17_{1}	18_{1}
19_{1}	20_{1}	21_{0}	*22*	*23*	24_{1}	25_{1}	26_{1}	27_{9}	28_{1}	29_{2}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{1}	36_{2}
37_{1}	38_{1}	39_{0}	*40*	*41*	42_{0}	43_{1}	44_{0}	45_{1}	46_{1}	47_{0}	48_{1}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{1}
55_{2}	56_{1}	57_{1}	*58*	*59*	60_{1}	61_{1}	62_{3}	63_{2}	64_{9}	65_{1}	66_{1}	*67*	*68*	69_{1}	70_{0}	71_{2}	72_{1}
73_{1}	74_{0}	75_{0}	*76*	*77*	78_{0}	79_{0}	80_{1}	81_{1}	82_{0}	83_{2}	84_{0}	*85*	*86*	87_{0}	88_{2}	89_{1}	90_{3}
91_{2}	92_{2}	93_{1}	*94*	*95*	96_{0}	97_{1}	98_{1}	99_{3}	100_{1}	101_{1}	102_{0}	*103*	*104*	105_{0}	106_{1}	107_{0}	108_{0}
109_{1}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{0}	116_{1}	117_{1}	118_{3}	119_{1}	120_{1}	*121*	*122*	123_{1}	124_{1}	125_{9}	126_{2}

(*complete up to* $|x|, |y|, |z| \leq 7$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{9}	2_{2}	3_{2}	*4*	*5*	6_{1}	7_{1}	8_{9}	9_{1}	10_{2}	11_{1}	12_{0}	*13*	*14*	15_{1}	16_{1}	17_{1}	18_{1}
19_{1}	20_{1}	21_{0}	*22*	*23*	24_{1}	25_{1}	26_{1}	27_{10}	28_{1}	29_{2}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{1}	36_{2}
37_{1}	38_{1}	39_{0}	*40*	*41*	42_{0}	43_{1}	44_{1}	45_{1}	46_{1}	47_{1}	48_{1}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{1}
55_{2}	56_{1}	57_{1}	*58*	*59*	60_{1}	61_{1}	62_{3}	63_{2}	64_{10}	65_{1}	66_{1}	*67*	*68*	69_{1}	70_{0}	71_{2}	72_{1}
73_{1}	74_{0}	75_{0}	*76*	*77*	78_{0}	79_{0}	80_{2}	81_{1}	82_{0}	83_{2}	84_{0}	*85*	*86*	87_{0}	88_{2}	89_{1}	90_{3}
91_{2}	92_{2}	93_{1}	*94*	*95*	96_{0}	97_{1}	98_{1}	99_{3}	100_{1}	101_{1}	102_{0}	*103*	*104*	105_{1}	106_{1}	107_{0}	108_{0}
109_{1}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{0}	116_{1}	117_{1}	118_{3}	119_{1}	120_{1}	*121*	*122*	123_{1}	124_{1}	125_{10}	126_{2}

(*complete up to* $|x|, |y|, |z| \leq 8$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{11}	2_{2}	3_{2}	*4*	*5*	6_{1}	7_{1}	8_{10}	9_{1}	10_{2}	11_{1}	12_{0}	*13*	*14*	15_{1}	16_{1}	17_{1}	18_{1}
19_{1}	20_{1}	21_{0}	*22*	*23*	24_{1}	25_{1}	26_{1}	27_{11}	28_{1}	29_{2}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{1}	36_{2}
37_{1}	38_{1}	39_{0}	*40*	*41*	42_{0}	43_{2}	44_{1}	45_{1}	46_{1}	47_{1}	48_{1}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{1}
55_{2}	56_{1}	57_{1}	*58*	*59*	60_{1}	61_{1}	62_{3}	63_{2}	64_{11}	65_{1}	66_{1}	*67*	*68*	69_{1}	70_{0}	71_{2}	72_{1}
73_{1}	74_{0}	75_{0}	*76*	*77*	78_{0}	79_{0}	80_{2}	81_{1}	82_{0}	83_{2}	84_{0}	*85*	*86*	87_{0}	88_{2}	89_{1}	90_{3}
91_{2}	92_{3}	93_{1}	*94*	*95*	96_{0}	97_{1}	98_{1}	99_{3}	100_{1}	101_{1}	102_{0}	*103*	*104*	105_{1}	106_{1}	107_{0}	108_{0}
109_{1}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{0}	116_{1}	117_{1}	118_{3}	119_{1}	120_{1}	*121*	*122*	123_{1}	124_{1}	125_{11}	126_{3}

(*complete up to* $|x|, |y|, |z| \leq 9$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{12}	2_{2}	3_{2}	*4*	*5*	6_{1}	7_{1}	8_{11}	9_{1}	10_{2}	11_{1}	12_{0}	*13*	*14*	15_{1}	16_{1}	17_{1}	18_{1}
19_{1}	20_{1}	21_{0}	*22*	*23*	24_{2}	25_{1}	26_{1}	27_{12}	28_{1}	29_{2}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{1}	36_{2}
37_{1}	38_{1}	39_{0}	*40*	*41*	42_{0}	43_{2}	44_{1}	45_{1}	46_{1}	47_{1}	48_{1}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{1}
55_{3}	56_{1}	57_{1}	*58*	*59*	60_{1}	61_{1}	62_{3}	63_{2}	64_{12}	65_{1}	66_{1}	*67*	*68*	69_{1}	70_{0}	71_{2}	72_{2}
73_{1}	74_{0}	75_{0}	*76*	*77*	78_{0}	79_{0}	80_{2}	81_{1}	82_{0}	83_{2}	84_{0}	*85*	*86*	87_{0}	88_{2}	89_{1}	90_{3}
91_{2}	92_{3}	93_{1}	*94*	*95*	96_{0}	97_{1}	98_{1}	99_{3}	100_{1}	101_{1}	102_{0}	*103*	*104*	105_{1}	106_{1}	107_{0}	108_{0}
109_{1}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{0}	116_{1}	117_{1}	118_{3}	119_{1}	120_{1}	*121*	*122*	123_{1}	124_{1}	125_{12}	126_{3}

(*complete up to* $|x|, |y|, |z| \leq 10$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{23}	2_{2}	3_{2}	*4*	*5*	6_{1}	7_{1}	8_{23}	9_{1}	10_{2}	11_{1}	12_{1}	*13*	*14*	15_{1}	16_{2}	17_{1}	18_{1}
19_{2}	20_{1}	21_{1}	*22*	*23*	24_{2}	25_{1}	26_{1}	27_{23}	28_{2}	29_{3}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{2}	36_{2}
37_{1}	38_{1}	39_{0}	*40*	*41*	42_{0}	43_{3}	44_{1}	45_{1}	46_{1}	47_{1}	48_{1}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{2}
55_{3}	56_{1}	57_{1}	*58*	*59*	60_{1}	61_{1}	62_{3}	63_{2}	64_{22}	65_{1}	66_{1}	*67*	*68*	69_{1}	70_{0}	71_{2}	72_{2}
73_{1}	74_{0}	75_{0}	*76*	*77*	78_{0}	79_{0}	80_{2}	81_{3}	82_{1}	83_{2}	84_{0}	*85*	*86*	87_{0}	88_{3}	89_{1}	90_{4}
91_{2}	92_{3}	93_{1}	*94*	*95*	96_{0}	97_{1}	98_{2}	99_{3}	100_{1}	101_{1}	102_{0}	*103*	*104*	105_{1}	106_{1}	107_{0}	108_{0}
109_{2}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{2}	116_{1}	117_{1}	118_{3}	119_{2}	120_{1}	*121*	*122*	123_{1}	124_{1}	125_{22}	126_{4}

(*complete up to* $|x|, |y|, |z| \leq 20$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{33}	2_{2}	3_{2}	*4*	*5*	6_{1}	7_{1}	8_{34}	9_{1}	10_{2}	11_{1}	12_{1}	*13*	*14*	15_{1}	16_{2}	17_{1}	18_{1}
19_{2}	20_{1}	21_{1}	*22*	*23*	24_{2}	25_{1}	26_{1}	27_{34}	28_{2}	29_{3}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{2}	36_{2}
37_{1}	38_{2}	39_{0}	*40*	*41*	42_{0}	43_{3}	44_{1}	45_{1}	46_{2}	47_{1}	48_{1}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{3}
55_{4}	56_{2}	57_{1}	*58*	*59*	60_{1}	61_{1}	62_{3}	63_{2}	64_{33}	65_{1}	66_{1}	*67*	*68*	69_{2}	70_{1}	71_{4}	72_{3}
73_{2}	74_{0}	75_{0}	*76*	*77*	78_{0}	79_{0}	80_{2}	81_{3}	82_{1}	83_{4}	84_{0}	*85*	*86*	87_{0}	88_{3}	89_{1}	90_{5}
91_{2}	92_{3}	93_{1}	*94*	*95*	96_{1}	97_{2}	98_{2}	99_{3}	100_{1}	101_{1}	102_{0}	*103*	*104*	105_{1}	106_{2}	107_{0}	108_{0}
109_{2}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{2}	116_{1}	117_{1}	118_{3}	119_{2}	120_{1}	*121*	*122*	123_{1}	124_{1}	125_{32}	126_{4}

(*complete up to* $|x|, |y|, |z| \leq 30$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{43}	2_{2}	3_{2}	*4*	*5*	6_{1}	7_{1}	8_{45}	9_{1}	10_{2}	11_{1}	12_{1}	*13*	*14*	15_{1}	16_{2}	17_{1}	18_{1}
19_{2}	20_{1}	21_{1}	*22*	*23*	24_{2}	25_{1}	26_{1}	27_{45}	28_{2}	29_{3}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{2}	36_{2}
37_{1}	38_{2}	39_{0}	*40*	*41*	42_{0}	43_{3}	44_{1}	45_{1}	46_{2}	47_{2}	48_{2}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{3}
55_{4}	56_{2}	57_{2}	*58*	*59*	60_{1}	61_{1}	62_{4}	63_{2}	64_{45}	65_{1}	66_{1}	*67*	*68*	69_{2}	70_{1}	71_{5}	72_{3}
73_{2}	74_{0}	75_{0}	*76*	*77*	78_{0}	79_{1}	80_{2}	81_{3}	82_{1}	83_{4}	84_{0}	*85*	*86*	87_{0}	88_{3}	89_{1}	90_{5}
91_{2}	92_{3}	93_{1}	*94*	*95*	96_{1}	97_{2}	98_{2}	99_{4}	100_{1}	101_{1}	102_{0}	*103*	*104*	105_{1}	106_{2}	107_{0}	108_{0}
109_{2}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{2}	116_{1}	117_{1}	118_{3}	119_{2}	120_{1}	*121*	*122*	123_{2}	124_{1}	125_{42}	126_{4}

(*complete up to* $|x|, |y|, |z| \leq 40$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{53}	2_{3}	3_{2}	*4*	*5*	6_{1}	7_{1}	8_{56}	9_{1}	10_{2}	11_{1}	12_{1}	*13*	*14*	15_{2}	16_{2}	17_{1}	18_{1}
19_{2}	20_{1}	21_{1}	*22*	*23*	24_{2}	25_{1}	26_{1}	27_{56}	28_{2}	29_{3}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{2}	36_{2}
37_{1}	38_{2}	39_{0}	*40*	*41*	42_{0}	43_{3}	44_{1}	45_{1}	46_{2}	47_{2}	48_{2}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{3}
55_{4}	56_{3}	57_{2}	*58*	*59*	60_{1}	61_{1}	62_{5}	63_{2}	64_{56}	65_{1}	66_{1}	*67*	*68*	69_{2}	70_{1}	71_{5}	72_{3}
73_{3}	74_{0}	75_{0}	*76*	*77*	78_{0}	79_{1}	80_{2}	81_{3}	82_{1}	83_{5}	84_{0}	*85*	*86*	87_{0}	88_{3}	89_{1}	90_{6}
91_{2}	92_{3}	93_{1}	*94*	*95*	96_{1}	97_{2}	98_{2}	99_{4}	100_{1}	101_{1}	102_{0}	*103*	*104*	105_{1}	106_{2}	107_{0}	108_{0}
109_{3}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{2}	116_{1}	117_{1}	118_{4}	119_{2}	120_{1}	*121*	*122*	123_{2}	124_{1}	125_{53}	126_{4}

(*complete up to* $|x|, |y|, |z| \leq 50$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{63}	2_{3}	3_{2}	*4*	*5*	6_{1}	7_{1}	8_{66}	9_{1}	10_{2}	11_{1}	12_{1}	*13*	*14*	15_{2}	16_{2}	17_{2}	18_{1}
19_{2}	20_{2}	21_{1}	*22*	*23*	24_{2}	25_{1}	26_{1}	27_{67}	28_{3}	29_{3}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{2}	36_{2}
37_{2}	38_{2}	39_{0}	*40*	*41*	42_{0}	43_{4}	44_{1}	45_{1}	46_{2}	47_{2}	48_{2}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{3}
55_{4}	56_{3}	57_{2}	*58*	*59*	60_{1}	61_{1}	62_{5}	63_{2}	64_{66}	65_{1}	66_{1}	*67*	*68*	69_{2}	70_{1}	71_{5}	72_{3}
73_{3}	74_{0}	75_{0}	*76*	*77*	78_{1}	79_{1}	80_{2}	81_{3}	82_{1}	83_{5}	84_{0}	*85*	*86*	87_{0}	88_{3}	89_{1}	90_{7}
91_{2}	92_{3}	93_{1}	*94*	*95*	96_{1}	97_{2}	98_{2}	99_{4}	100_{1}	101_{1}	102_{0}	*103*	*104*	105_{1}	106_{2}	107_{1}	108_{0}
109_{3}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{2}	116_{1}	117_{1}	118_{5}	119_{2}	120_{1}	*121*	*122*	123_{2}	124_{1}	125_{64}	126_{4}

(*complete up to* $|x|, |y|, |z| \leq 60$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{73}	2_{3}	3_{2}	*4*	*5*	6_{2}	7_{1}	8_{76}	9_{1}	10_{2}	11_{1}	12_{1}	*13*	*14*	15_{2}	16_{2}	17_{2}	18_{1}
19_{2}	20_{2}	21_{1}	*22*	*23*	24_{2}	25_{1}	26_{1}	27_{77}	28_{3}	29_{3}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{2}	36_{2}
37_{2}	38_{2}	39_{0}	*40*	*41*	42_{0}	43_{4}	44_{1}	45_{1}	46_{2}	47_{3}	48_{2}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{3}
55_{4}	56_{3}	57_{2}	*58*	*59*	60_{1}	61_{1}	62_{5}	63_{4}	64_{77}	65_{1}	66_{1}	*67*	*68*	69_{2}	70_{2}	71_{5}	72_{3}
73_{3}	74_{0}	75_{0}	*76*	*77*	78_{1}	79_{1}	80_{2}	81_{3}	82_{1}	83_{5}	84_{0}	*85*	*86*	87_{0}	88_{3}	89_{1}	90_{7}
91_{2}	92_{3}	93_{1}	*94*	*95*	96_{1}	97_{2}	98_{2}	99_{4}	100_{1}	101_{1}	102_{0}	*103*	*104*	105_{1}	106_{2}	107_{1}	108_{0}
109_{3}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{2}	116_{1}	117_{1}	118_{5}	119_{2}	120_{1}	*121*	*122*	123_{2}	124_{1}	125_{74}	126_{4}

(*complete up to* $|x|, |y|, |z| \leq 70$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{83}	2_{3}	3_{2}	*4*	*5*	6_{2}	7_{1}	8_{86}	9_{1}	10_{2}	11_{1}	12_{1}	*13*	*14*	15_{2}	16_{2}	17_{2}	18_{1}
19_{3}	20_{2}	21_{1}	*22*	*23*	24_{2}	25_{1}	26_{1}	27_{87}	28_{3}	29_{3}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{2}	36_{3}
37_{2}	38_{2}	39_{0}	*40*	*41*	42_{0}	43_{4}	44_{1}	45_{1}	46_{2}	47_{3}	48_{2}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{3}
55_{4}	56_{3}	57_{2}	*58*	*59*	60_{1}	61_{1}	62_{5}	63_{4}	64_{87}	65_{1}	66_{1}	*67*	*68*	69_{2}	70_{2}	71_{5}	72_{3}
73_{3}	74_{0}	75_{0}	*76*	*77*	78_{1}	79_{2}	80_{2}	81_{3}	82_{1}	83_{5}	84_{0}	*85*	*86*	87_{0}	88_{3}	89_{1}	90_{8}
91_{2}	92_{3}	93_{1}	*94*	*95*	96_{1}	97_{2}	98_{2}	99_{4}	100_{1}	101_{1}	102_{0}	*103*	*104*	105_{1}	106_{2}	107_{1}	108_{0}
109_{3}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{2}	116_{1}	117_{1}	118_{5}	119_{2}	120_{1}	*121*	*122*	123_{2}	124_{1}	125_{85}	126_{4}

(*complete up to* $|x|, |y|, |z| \leq 80$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{93}	2_{3}	3_{2}	*4*	*5*	6_{2}	7_{1}	8_{97}	9_{1}	10_{2}	11_{1}	12_{1}	*13*	*14*	15_{2}	16_{2}	17_{2}	18_{1}
19_{3}	20_{2}	21_{2}	*22*	*23*	24_{2}	25_{1}	26_{1}	27_{97}	28_{3}	29_{3}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{2}	36_{3}
37_{2}	38_{2}	39_{0}	*40*	*41*	42_{0}	43_{4}	44_{1}	45_{1}	46_{2}	47_{3}	48_{2}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{3}
55_{4}	56_{3}	57_{2}	*58*	*59*	60_{1}	61_{1}	62_{5}	63_{4}	64_{98}	65_{1}	66_{1}	*67*	*68*	69_{2}	70_{2}	71_{5}	72_{3}
73_{3}	74_{0}	75_{0}	*76*	*77*	78_{1}	79_{2}	80_{2}	81_{3}	82_{1}	83_{5}	84_{0}	*85*	*86*	87_{0}	88_{3}	89_{1}	90_{8}
91_{2}	92_{3}	93_{1}	*94*	*95*	96_{1}	97_{2}	98_{2}	99_{4}	100_{1}	101_{2}	102_{0}	*103*	*104*	105_{1}	106_{2}	107_{1}	108_{0}
109_{3}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{2}	116_{1}	117_{1}	118_{5}	119_{2}	120_{1}	*121*	*122*	123_{2}	124_{1}	125_{95}	126_{4}

(*complete up to* $|x|, |y|, |z| \leq 90$)

Historical progression

Table: A list of integers k from 1 to 126. The bottom right value of the k-th cell gives the number of integer solutions we've computed so far to $x^3 + y^3 + z^3 = k$ with $x \leq y \leq z$.

1_{103}	2_{3}	3_{2}	*4*	*5*	6_{2}	7_{1}	8_{107}	9_{1}	10_{2}	11_{1}	12_{1}	*13*	*14*	15_{2}	16_{3}	17_{2}	18_{1}
19_{4}	20_{2}	21_{2}	*22*	*23*	24_{2}	25_{1}	26_{1}	27_{107}	28_{3}	29_{3}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{2}	36_{3}
37_{2}	38_{2}	39_{0}	*40*	*41*	42_{0}	43_{4}	44_{1}	45_{1}	46_{2}	47_{3}	48_{2}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{3}
55_{4}	56_{3}	57_{2}	*58*	*59*	60_{1}	61_{1}	62_{5}	63_{4}	64_{108}	65_{1}	66_{1}	*67*	*68*	69_{2}	70_{2}	71_{5}	72_{3}
73_{3}	74_{0}	75_{0}	*76*	*77*	78_{1}	79_{2}	80_{2}	81_{3}	82_{1}	83_{5}	84_{0}	*85*	*86*	87_{0}	88_{3}	89_{1}	90_{9}
91_{2}	92_{3}	93_{1}	*94*	*95*	96_{1}	97_{2}	98_{2}	99_{4}	100_{1}	101_{2}	102_{0}	*103*	*104*	105_{1}	106_{2}	107_{1}	108_{0}
109_{3}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{2}	116_{1}	117_{1}	118_{6}	119_{2}	120_{2}	*121*	*122*	123_{2}	124_{1}	125_{105}	126_{4}

(*complete up to* $|x|, |y|, |z| \leq 100$)

Historical progression

1908: Verebryusov finds parametrisation for $k = 2$,

$$(1 + 6c^3)^3 + (1 - 6c^3)^3 + (-6c^2)^3 = 2.$$

1_{103}	2_{∞}	3_{2}	*4*	*5*	6_{2}	7_{1}	8_{107}	9_{1}	10_{2}	11_{1}	12_{1}	*13*	*14*	15_{2}	16_{∞}	17_{2}	18_{1}
19_{4}	20_{2}	21_{2}	*22*	*23*	24_{2}	25_{1}	26_{1}	27_{107}	28_{3}	29_{3}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{2}	36_{3}
37_{2}	38_{2}	39_{0}	*40*	*41*	42_{0}	43_{4}	44_{1}	45_{1}	46_{2}	47_{3}	48_{2}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{∞}
55_{4}	56_{3}	57_{2}	*58*	*59*	60_{1}	61_{1}	62_{5}	63_{4}	64_{108}	65_{1}	66_{1}	*67*	*68*	69_{2}	70_{2}	71_{5}	72_{3}
73_{3}	74_{0}	75_{0}	*76*	*77*	78_{1}	79_{2}	80_{2}	81_{3}	82_{1}	83_{5}	84_{0}	*85*	*86*	87_{0}	88_{3}	89_{1}	90_{9}
91_{2}	92_{3}	93_{1}	*94*	*95*	96_{1}	97_{2}	98_{2}	99_{4}	100_{1}	101_{2}	102_{0}	*103*	*104*	105_{1}	106_{2}	107_{1}	108_{0}
109_{3}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{2}	116_{1}	117_{1}	118_{6}	119_{2}	120_{2}	*121*	*122*	123_{2}	124_{1}	125_{105}	126_{4}

(*complete up to* $|x|, |y|, |z| \leq 100$)

Historical progression

Dec 1935: Mahler finds (non-trivial) parametrisation for $k = 1$,

$$(9b^4)^3 + (3b - 9b^4)^3 + (1 - 9b^3)^3 = 1.$$

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{2}	7_{1}	8_{∞}	9_{1}	10_{2}	11_{1}	12_{1}	*13*	*14*	15_{2}	16_{∞}	17_{2}	18_{1}
19_{4}	20_{2}	21_{2}	*22*	*23*	24_{2}	25_{1}	26_{1}	27_{∞}	28_{3}	29_{3}	30_{0}	*31*	*32*	33_{0}	34_{3}	35_{2}	36_{3}
37_{2}	38_{2}	39_{0}	*40*	*41*	42_{0}	43_{4}	44_{1}	45_{1}	46_{2}	47_{3}	48_{2}	*49*	*50*	51_{0}	52_{0}	53_{2}	54_{∞}
55_{4}	56_{3}	57_{2}	*58*	*59*	60_{1}	61_{1}	62_{5}	63_{4}	64_{∞}	65_{1}	66_{1}	*67*	*68*	69_{2}	70_{2}	71_{5}	72_{3}
73_{3}	74_{0}	75_{0}	*76*	*77*	78_{1}	79_{2}	80_{2}	81_{3}	82_{1}	83_{5}	84_{0}	*85*	*86*	87_{0}	88_{3}	89_{1}	90_{9}
91_{2}	92_{3}	93_{1}	*94*	*95*	96_{1}	97_{2}	98_{2}	99_{4}	100_{1}	101_{2}	102_{0}	*103*	*104*	105_{1}	106_{2}	107_{1}	108_{0}
109_{3}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{2}	116_{1}	117_{1}	118_{6}	119_{2}	120_{2}	*121*	*122*	123_{2}	124_{1}	125_{∞}	126_{4}

(*complete up to* $|x|, |y|, |z| \leq 100$)

Historical progression

May 1954: Miller, Woollett uses the EDSAC to compute all solutions to $x^3 + y^3 + z^3 = k$ for $k \leq 100$ and $|x|, |y|, |z| \leq 3200$.

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{4}	7_{3}	8_{∞}	9_{2}	10_{4}	11_{4}	12_{1}	*13*	*14*	15_{3}	16_{∞}	17_{5}	18_{4}
19_{4}	20_{7}	21_{4}	*22*	*23*	24_{2}	25_{3}	26_{4}	27_{∞}	28_{5}	29_{4}	30_{0}	*31*	*32*	33_{0}	34_{8}	35_{3}	36_{4}
37_{3}	38_{2}	39_{0}	*40*	*41*	42_{0}	43_{5}	44_{1}	45_{3}	46_{3}	47_{4}	48_{5}	*49*	*50*	51_{1}	52_{0}	53_{5}	54_{∞}
55_{11}	56_{6}	57_{7}	*58*	*59*	60_{2}	61_{2}	62_{7}	63_{5}	64_{∞}	65_{5}	66_{1}	*67*	*68*	69_{4}	70_{5}	71_{10}	72_{4}
73_{4}	74_{0}	75_{0}	*76*	*77*	78_{2}	79_{3}	80_{4}	81_{5}	82_{2}	83_{13}	84_{0}	*85*	*86*	87_{0}	88_{7}	89_{3}	90_{14}
91_{5}	92_{8}	93_{2}	*94*	*95*	96_{1}	97_{7}	98_{4}	99_{9}	100_{3}	101_{2}	102_{0}	*103*	*104*	105_{1}	106_{2}	107_{1}	108_{0}
109_{3}	110_{0}	111_{0}	*112*	*113*	114_{0}	115_{2}	116_{1}	117_{1}	118_{6}	119_{2}	120_{2}	*121*	*122*	123_{2}	124_{1}	125_{∞}	126_{4}

(*complete up to* $|x|, |y|, |z| \leq 100$)

Historical progression

Nov 1963: Gardiner, Lazarus, Stein compute all solutions to $x^3 + y^3 + z^3 = k$ for $k < 1000$ and $|x|, |y|, |z| \leq 2^{16}$.

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{5}	7_{3}	8_{∞}	9_{3}	10_{4}	11_{5}	12_{1}	*13*	*14*	15_{3}	16_{∞}	17_{7}	18_{7}
19_{4}	20_{8}	21_{5}	*22*	*23*	24_{2}	25_{4}	26_{5}	27_{∞}	28_{6}	29_{7}	30_{0}	*31*	*32*	33_{0}	34_{10}	35_{7}	36_{6}
37_{3}	38_{3}	39_{0}	*40*	*41*	42_{0}	43_{7}	44_{1}	45_{4}	46_{3}	47_{4}	48_{6}	*49*	*50*	51_{1}	52_{0}	53_{5}	54_{∞}
55_{14}	56_{6}	57_{10}	*58*	*59*	60_{3}	61_{2}	62_{10}	63_{8}	64_{∞}	65_{8}	66_{1}	*67*	*68*	69_{4}	70_{6}	71_{12}	72_{5}
73_{7}	74_{0}	75_{0}	*76*	*77*	78_{2}	79_{3}	80_{4}	81_{5}	82_{3}	83_{16}	84_{0}	*85*	*86*	87_{1}	88_{10}	89_{3}	90_{19}
91_{7}	92_{11}	93_{2}	*94*	*95*	96_{2}	97_{9}	98_{4}	99_{18}	100_{3}	101_{6}	102_{1}	*103*	*104*	105_{1}	106_{5}	107_{3}	108_{4}
109_{12}	110_{0}	111_{4}	*112*	*113*	114_{0}	115_{3}	116_{2}	117_{3}	118_{16}	119_{4}	120_{5}	*121*	*122*	123_{2}	124_{2}	125_{∞}	126_{10}

(*complete up to* $|x|, |y|, |z| \leq 65\,536$)

Historical progression

Sep 1990: Heath-Brown presents an $O_k(N \log N)$ algorithm for searching for solutions $|x|, |y|, |z| \leq N$ by considering the arithmetic of $\mathbb{Q}(\sqrt[3]{k})$.

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{5}	7_{3}	8_{∞}	9_{3}	10_{4}	11_{5}	12_{1}	*13*	*14*	15_{3}	16_{∞}	17_{7}	18_{7}
19_{4}	20_{8}	21_{5}	*22*	*23*	24_{2}	25_{4}	26_{5}	27_{∞}	28_{6}	29_{7}	30_{0}	*31*	*32*	33_{0}	34_{10}	35_{7}	36_{6}
37_{3}	38_{3}	39_{0}	*40*	*41*	42_{0}	43_{7}	44_{1}	45_{4}	46_{3}	47_{4}	48_{6}	*49*	*50*	51_{1}	52_{0}	53_{5}	54_{∞}
55_{14}	56_{6}	57_{10}	*58*	*59*	60_{3}	61_{2}	62_{10}	63_{8}	64_{∞}	65_{8}	66_{1}	*67*	*68*	69_{4}	70_{6}	71_{12}	72_{5}
73_{7}	74_{0}	75_{0}	*76*	*77*	78_{2}	79_{3}	80_{4}	81_{5}	82_{3}	83_{16}	84_{0}	*85*	*86*	87_{1}	88_{10}	89_{3}	90_{19}
91_{7}	92_{11}	93_{2}	*94*	*95*	96_{2}	97_{9}	98_{4}	99_{18}	100_{3}	101_{6}	102_{1}	*103*	*104*	105_{1}	106_{5}	107_{3}	108_{4}
109_{12}	110_{0}	111_{4}	*112*	*113*	114_{0}	115_{3}	116_{2}	117_{3}	118_{16}	119_{4}	120_{5}	*121*	*122*	123_{2}	124_{2}	125_{∞}	126_{10}

(*complete up to* $|x|, |y|, |z| \leq 65\,536$)

Historical progression

Jun 1991: Payne, Vaserstein compute all solutions to $x^3 + y^3 + z^3 = k$ for $k \leq 12$ up to $|x|, |y|, |z| \leq 10^5$.

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{5}	7_{3}	8_{∞}	9_{3}	10_{4}	11_{5}	12_{1}	*13*	*14*	15_{3}	16_{∞}	17_{7}	18_{7}
19_{4}	20_{8}	21_{5}	*22*	*23*	24_{2}	25_{4}	26_{5}	27_{∞}	28_{6}	29_{7}	30_{0}	*31*	*32*	33_{0}	34_{10}	35_{7}	36_{6}
37_{3}	38_{3}	39_{0}	*40*	*41*	42_{0}	43_{7}	44_{1}	45_{4}	46_{3}	47_{4}	48_{6}	*49*	*50*	51_{1}	52_{0}	53_{5}	54_{∞}
55_{14}	56_{6}	57_{10}	*58*	*59*	60_{3}	61_{2}	62_{10}	63_{8}	64_{∞}	65_{8}	66_{1}	*67*	*68*	69_{4}	70_{6}	71_{12}	72_{5}
73_{7}	74_{0}	75_{0}	*76*	*77*	78_{2}	79_{3}	80_{4}	81_{5}	82_{3}	83_{16}	84_{0}	*85*	*86*	87_{1}	88_{10}	89_{3}	90_{19}
91_{7}	92_{11}	93_{2}	*94*	*95*	96_{2}	97_{9}	98_{4}	99_{18}	100_{3}	101_{6}	102_{1}	*103*	*104*	105_{1}	106_{5}	107_{3}	108_{4}
109_{12}	110_{0}	111_{4}	*112*	*113*	114_{0}	115_{3}	116_{2}	117_{3}	118_{16}	119_{4}	120_{5}	*121*	*122*	123_{2}	124_{2}	125_{∞}	126_{10}

(*complete up to* $|x|, |y|, |z| \leq 65\,536$)

Historical progression

Mar 1992: Heath-Brown, Lioen, te Riele find first solution for $k = 39$,

$$134\,476^3 + 117\,367^3 + (-159\,380)^3 = 39.$$

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{5}	7_{3}	8_{∞}	9_{3}	10_{4}	11_{5}	12_{1}	*13*	*14*	15_{3}	16_{∞}	17_{7}	18_{7}
19_{4}	20_{16}	21_{5}	*22*	*23*	24_{2}	25_{4}	26_{5}	27_{∞}	28_{6}	29_{7}	30_{0}	*31*	*32*	33_{0}	34_{10}	35_{7}	36_{6}
37_{3}	38_{3}	39_{1}	*40*	*41*	42_{0}	43_{7}	44_{1}	45_{4}	46_{3}	47_{4}	48_{6}	*49*	*50*	51_{1}	52_{0}	53_{5}	54_{∞}
55_{14}	56_{6}	57_{10}	*58*	*59*	60_{3}	61_{2}	62_{10}	63_{8}	64_{∞}	65_{8}	66_{1}	*67*	*68*	69_{4}	70_{6}	71_{12}	72_{5}
73_{7}	74_{0}	75_{0}	*76*	*77*	78_{2}	79_{3}	80_{4}	81_{5}	82_{3}	83_{16}	84_{0}	*85*	*86*	87_{1}	88_{10}	89_{3}	90_{19}
91_{7}	92_{11}	93_{2}	*94*	*95*	96_{2}	97_{9}	98_{4}	99_{18}	100_{3}	101_{6}	102_{1}	*103*	*104*	105_{1}	106_{5}	107_{3}	108_{4}
109_{12}	110_{0}	111_{4}	*112*	*113*	114_{0}	115_{3}	116_{2}	117_{3}	118_{16}	119_{4}	120_{5}	*121*	*122*	123_{2}	124_{2}	125_{∞}	126_{10}

(*complete up to* $|x|, |y|, |z| \le 65\,536$)

Historical progression

Jul 1992: Conn, Vaserstein find first solution for $k = 84$,

$$41\,639\,611^3 + (-41\,531\,726)^3 + (-8\,241\,191)^3 = 84.$$

1_{∞}	2_{∞}	3_2	*4*	*5*	6_5	7_3	8_{∞}	9_3	10_4	11_5	12_1	*13*	*14*	15_3	16_{∞}	17_7	18_7
19_4	20_{16}	21_5	*22*	*23*	24_2	25_4	26_5	27_{∞}	28_6	29_7	30_0	*31*	*32*	33_0	34_{10}	35_7	36_6
37_3	38_3	39_1	*40*	*41*	42_0	43_7	44_1	45_4	46_3	47_4	48_6	*49*	*50*	51_1	52_0	53_5	54_{∞}
55_{14}	56_6	57_{10}	*58*	*59*	60_3	61_2	62_{10}	63_8	64_{∞}	65_8	66_1	*67*	*68*	69_4	70_6	71_{12}	72_5
73_7	74_0	75_0	*76*	*77*	78_2	79_3	80_4	81_5	82_3	83_{16}	84_1	*85*	*86*	87_1	88_{10}	89_3	90_{19}
91_7	92_{11}	93_2	*94*	*95*	96_2	97_9	98_4	99_{18}	100_3	101_6	102_1	*103*	*104*	105_1	106_5	107_3	108_4
109_{12}	110_0	111_4	*112*	*113*	114_0	115_3	116_2	117_3	118_{16}	119_4	120_5	*121*	*122*	123_2	124_2	125_{∞}	126_{10}

(*complete up to* $|x|, |y|, |z| \leq 65\,536$)

Historical progression

Apr 1994: Koyama computes all solutions to $x^3 + y^3 + z^3 = k$ for $k \leq 1000$ up to $|x|, |y|, |z| \leq 2^{21} - 1$.

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{5}	7_{4}	8_{∞}	9_{3}	10_{5}	11_{6}	12_{1}	*13*	*14*	15_{5}	16_{∞}	17_{9}	18_{10}
19_{6}	20_{16}	21_{8}	*22*	*23*	24_{2}	25_{4}	26_{6}	27_{∞}	28_{7}	29_{11}	30_{0}	*31*	*32*	33_{0}	34_{12}	35_{10}	36_{8}
37_{4}	38_{3}	39_{1}	*40*	*41*	42_{0}	43_{13}	44_{4}	45_{8}	46_{7}	47_{8}	48_{7}	*49*	*50*	51_{2}	52_{0}	53_{10}	54_{∞}
55_{16}	56_{9}	57_{13}	*58*	*59*	60_{4}	61_{2}	62_{12}	63_{11}	64_{∞}	65_{8}	66_{2}	*67*	*68*	69_{5}	70_{7}	71_{14}	72_{5}
73_{8}	74_{0}	75_{0}	*76*	*77*	78_{2}	79_{3}	80_{8}	81_{7}	82_{5}	83_{21}	84_{1}	*85*	*86*	87_{1}	88_{12}	89_{4}	90_{25}
91_{12}	92_{14}	93_{2}	*94*	*95*	96_{2}	97_{14}	98_{6}	99_{22}	100_{4}	101_{6}	102_{1}	*103*	*104*	105_{1}	106_{7}	107_{6}	108_{5}
109_{14}	110_{0}	111_{5}	*112*	*113*	114_{0}	115_{5}	116_{5}	117_{3}	118_{19}	119_{9}	120_{8}	*121*	*122*	123_{2}	124_{4}	125_{∞}	126_{13}

(*complete up to* $|x|, |y|, |z| \leq 2\,097\,151$)

Historical progression

Jul 1994: Bremner finds first solution for $k = 75$,

$$(-435\,203\,231)^3 + 4\,381\,159^3 + 435\,203\,083^3 = 75.$$

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{5}	7_{4}	8_{∞}	9_{3}	10_{5}	11_{6}	12_{1}	*13*	*14*	15_{5}	16_{∞}	17_{9}	18_{10}
19_{6}	20_{16}	21_{8}	*22*	*23*	24_{2}	25_{4}	26_{6}	27_{∞}	28_{7}	29_{11}	30_{0}	*31*	*32*	33_{0}	34_{12}	35_{10}	36_{8}
37_{4}	38_{3}	39_{1}	*40*	*41*	42_{0}	43_{13}	44_{4}	45_{8}	46_{7}	47_{8}	48_{7}	*49*	*50*	51_{2}	52_{0}	53_{10}	54_{∞}
55_{16}	56_{9}	57_{13}	*58*	*59*	60_{4}	61_{2}	62_{12}	63_{11}	64_{∞}	65_{8}	66_{2}	*67*	*68*	69_{5}	70_{7}	71_{14}	72_{5}
73_{8}	74_{0}	75_{1}	*76*	*77*	78_{2}	79_{3}	80_{8}	81_{7}	82_{5}	83_{21}	84_{1}	*85*	*86*	87_{1}	88_{12}	89_{4}	90_{25}
91_{12}	92_{14}	93_{2}	*94*	*95*	96_{2}	97_{14}	98_{6}	99_{22}	100_{4}	101_{6}	102_{1}	*103*	*104*	105_{1}	106_{7}	107_{6}	108_{5}
109_{14}	110_{0}	111_{5}	*112*	*113*	114_{0}	115_{5}	116_{5}	117_{3}	118_{19}	119_{9}	120_{8}	*121*	*122*	123_{2}	124_{4}	125_{∞}	126_{13}

(*complete up to* $|x|, |y|, |z| \leq 2\,097\,151$)

Historical progression

Mar 1995: Koyama extends his search for solutions for $k \leq 1000$ up to $|x|, |y|, |z| \leq 3\,414\,387$.

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{6}	7_{4}	8_{∞}	9_{3}	10_{5}	11_{6}	12_{1}	*13*	*14*	15_{5}	16_{∞}	17_{9}	18_{10}
19_{6}	20_{16}	21_{8}	*22*	*23*	24_{2}	25_{4}	26_{6}	27_{∞}	28_{8}	29_{11}	30_{0}	*31*	*32*	33_{0}	34_{13}	35_{11}	36_{8}
37_{4}	38_{3}	39_{1}	*40*	*41*	42_{0}	43_{13}	44_{4}	45_{8}	46_{7}	47_{8}	48_{7}	*49*	*50*	51_{3}	52_{0}	53_{11}	54_{∞}
55_{16}	56_{10}	57_{13}	*58*	*59*	60_{4}	61_{2}	62_{13}	63_{11}	64_{∞}	65_{8}	66_{2}	*67*	*68*	69_{5}	70_{9}	71_{14}	72_{5}
73_{8}	74_{0}	75_{1}	*76*	*77*	78_{2}	79_{3}	80_{8}	81_{7}	82_{5}	83_{21}	84_{1}	*85*	*86*	87_{1}	88_{12}	89_{4}	90_{25}
91_{12}	92_{14}	93_{2}	*94*	*95*	96_{2}	97_{15}	98_{6}	99_{25}	100_{4}	101_{6}	102_{1}	*103*	*104*	105_{1}	106_{7}	107_{6}	108_{5}
109_{14}	110_{0}	111_{5}	*112*	*113*	114_{0}	115_{6}	116_{5}	117_{3}	118_{19}	119_{9}	120_{9}	*121*	*122*	123_{2}	124_{4}	125_{∞}	126_{14}

(*complete up to* $|x|, |y|, |z| \leq 3\,414\,387$)

Historical progression

Sep 1995: Lukes finds first solution for $k = 110$,

$$(-16\,540\,291\,649)^3 + 109\,938\,919^3 + 16\,540\,290\,030^3 = 110.$$

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{6}	7_{4}	8_{∞}	9_{3}	10_{5}	11_{6}	12_{1}	*13*	*14*	15_{5}	16_{∞}	17_{9}	18_{10}
19_{6}	20_{16}	21_{8}	*22*	*23*	24_{2}	25_{4}	26_{6}	27_{∞}	28_{8}	29_{11}	30_{0}	*31*	*32*	33_{0}	34_{13}	35_{11}	36_{8}
37_{4}	38_{3}	39_{1}	*40*	*41*	42_{0}	43_{13}	44_{4}	45_{8}	46_{7}	47_{8}	48_{7}	*49*	*50*	51_{3}	52_{0}	53_{11}	54_{∞}
55_{16}	56_{10}	57_{13}	*58*	*59*	60_{4}	61_{2}	62_{13}	63_{11}	64_{∞}	65_{8}	66_{2}	*67*	*68*	69_{5}	70_{9}	71_{14}	72_{5}
73_{8}	74_{0}	75_{1}	*76*	*77*	78_{2}	79_{3}	80_{8}	81_{7}	82_{5}	83_{21}	84_{1}	*85*	*86*	87_{1}	88_{12}	89_{4}	90_{25}
91_{12}	92_{14}	93_{2}	*94*	*95*	96_{2}	97_{15}	98_{6}	99_{25}	100_{4}	101_{6}	102_{1}	*103*	*104*	105_{1}	106_{7}	107_{6}	108_{5}
109_{14}	110_{1}	111_{5}	*112*	*113*	114_{0}	115_{6}	116_{5}	117_{3}	118_{19}	119_{9}	120_{9}	*121*	*122*	123_{2}	124_{4}	125_{∞}	126_{14}

(*complete up to* $|x|, |y|, |z| \leq 3\,414\,387$)

Historical progression

Nov 1995: Koyama, Tsuruoka, Sekigawa compute all solutions for $k < 1000$ up to $|x|, |y|, |z| \leq 2 \cdot 10^7$.

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{6}	7_{7}	8_{∞}	9_{3}	10_{5}	11_{6}	12_{2}	*13*	*14*	15_{5}	16_{∞}	17_{9}	18_{10}
19_{7}	20_{16}	21_{8}	*22*	*23*	24_{2}	25_{4}	26_{6}	27_{∞}	28_{12}	29_{12}	30_{0}	*31*	*32*	33_{0}	34_{14}	35_{12}	36_{9}
37_{5}	38_{3}	39_{1}	*40*	*41*	42_{0}	43_{16}	44_{4}	45_{8}	46_{7}	47_{8}	48_{8}	*49*	*50*	51_{3}	52_{0}	53_{11}	54_{∞}
55_{17}	56_{13}	57_{15}	*58*	*59*	60_{4}	61_{2}	62_{14}	63_{13}	64_{∞}	65_{8}	66_{2}	*67*	*68*	69_{6}	70_{10}	71_{20}	72_{5}
73_{9}	74_{0}	75_{1}	*76*	*77*	78_{2}	79_{4}	80_{8}	81_{8}	82_{5}	83_{23}	84_{1}	*85*	*86*	87_{1}	88_{12}	89_{6}	90_{30}
91_{16}	92_{17}	93_{2}	*94*	*95*	96_{3}	97_{17}	98_{6}	99_{28}	100_{4}	101_{6}	102_{1}	*103*	*104*	105_{1}	106_{7}	107_{6}	108_{5}
109_{14}	110_{1}	111_{5}	*112*	*113*	114_{0}	115_{6}	116_{6}	117_{4}	118_{23}	119_{9}	120_{9}	*121*	*122*	123_{2}	124_{4}	125_{∞}	126_{16}

(*complete up to* $|x|, |y|, |z| \leq 2 \cdot 10^7$)

Historical progression

May 1996: Elkies presents a new method of finding rational points near the Fermat curve $X^3 + Y^3 = 1$ to get all solutions for $k \leq 1000$ up to $N = 10^7$.

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{6}	7_{7}	8_{∞}	9_{3}	10_{5}	11_{6}	12_{2}	*13*	*14*	15_{5}	16_{∞}	17_{9}	18_{10}
19_{7}	20_{16}	21_{8}	*22*	*23*	24_{2}	25_{4}	26_{6}	27_{∞}	28_{12}	29_{12}	30_{0}	*31*	*32*	33_{0}	34_{14}	35_{12}	36_{9}
37_{5}	38_{3}	39_{1}	*40*	*41*	42_{0}	43_{16}	44_{4}	45_{8}	46_{7}	47_{8}	48_{8}	*49*	*50*	51_{3}	52_{0}	53_{11}	54_{∞}
55_{17}	56_{13}	57_{15}	*58*	*59*	60_{4}	61_{2}	62_{14}	63_{13}	64_{∞}	65_{8}	66_{2}	*67*	*68*	69_{6}	70_{10}	71_{20}	72_{5}
73_{9}	74_{0}	75_{1}	*76*	*77*	78_{2}	79_{4}	80_{8}	81_{8}	82_{5}	83_{23}	84_{1}	*85*	*86*	87_{1}	88_{12}	89_{6}	90_{30}
91_{16}	92_{17}	93_{2}	*94*	*95*	96_{3}	97_{17}	98_{6}	99_{28}	100_{4}	101_{6}	102_{1}	*103*	*104*	105_{1}	106_{7}	107_{6}	108_{5}
109_{14}	110_{1}	111_{5}	*112*	*113*	114_{0}	115_{6}	116_{6}	117_{4}	118_{23}	119_{9}	120_{9}	*121*	*122*	123_{2}	124_{4}	125_{∞}	126_{16}

(*complete up to* $|x|, |y|, |z| \leq 2 \cdot 10^7$)

Historical progression

May 1997: Deep Blue defeats Garry Kasparov in chess.

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{6}	7_{7}	8_{∞}	9_{3}	10_{5}	11_{6}	12_{2}	*13*	*14*	15_{5}	16_{∞}	17_{9}	18_{10}
19_{7}	20_{16}	21_{8}	*22*	*23*	24_{2}	25_{4}	26_{6}	27_{∞}	28_{12}	29_{12}	30_{0}	*31*	*32*	33_{0}	34_{14}	35_{12}	36_{9}
37_{5}	38_{3}	39_{1}	*40*	*41*	42_{0}	43_{16}	44_{4}	45_{8}	46_{7}	47_{8}	48_{8}	*49*	*50*	51_{3}	52_{0}	53_{11}	54_{∞}
55_{17}	56_{13}	57_{15}	*58*	*59*	60_{4}	61_{2}	62_{14}	63_{13}	64_{∞}	65_{8}	66_{2}	*67*	*68*	69_{6}	70_{10}	71_{20}	72_{5}
73_{9}	74_{0}	75_{1}	*76*	*77*	78_{2}	79_{4}	80_{8}	81_{8}	82_{5}	83_{23}	84_{1}	*85*	*86*	87_{1}	88_{12}	89_{6}	90_{30}
91_{16}	92_{17}	93_{2}	*94*	*95*	96_{3}	97_{17}	98_{6}	99_{28}	100_{4}	101_{6}	102_{1}	*103*	*104*	105_{1}	106_{7}	107_{6}	108_{5}
109_{14}	110_{1}	111_{5}	*112*	*113*	114_{0}	115_{6}	116_{6}	117_{4}	118_{23}	119_{9}	120_{9}	*121*	*122*	123_{2}	124_{4}	125_{∞}	126_{16}

(*complete up to* $|x|, |y|, |z| \leq 2 \cdot 10^7$)

Historical progression

Jul 1999: Beck, Pine, Tarrant, Yarbrough Jensen find first solution for $k = 30$,

$$(-2\,218\,888\,517)^3 + (-283\,059\,965)^3 + 2\,220\,422\,932^3 = 30.$$

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{6}	7_{7}	8_{∞}	9_{3}	10_{5}	11_{6}	12_{2}	*13*	*14*	15_{5}	16_{∞}	17_{9}	18_{10}
19_{7}	20_{16}	21_{8}	*22*	*23*	24_{2}	25_{4}	26_{6}	27_{∞}	28_{12}	29_{12}	30_{1}	*31*	*32*	33_{0}	34_{14}	35_{12}	36_{9}
37_{5}	38_{3}	39_{1}	*40*	*41*	42_{0}	43_{16}	44_{4}	45_{8}	46_{7}	47_{8}	48_{8}	*49*	*50*	51_{3}	52_{0}	53_{11}	54_{∞}
55_{17}	56_{13}	57_{15}	*58*	*59*	60_{4}	61_{2}	62_{14}	63_{13}	64_{∞}	65_{8}	66_{2}	*67*	*68*	69_{6}	70_{10}	71_{20}	72_{5}
73_{9}	74_{0}	75_{1}	*76*	*77*	78_{2}	79_{4}	80_{8}	81_{8}	82_{5}	83_{23}	84_{1}	*85*	*86*	87_{1}	88_{12}	89_{6}	90_{30}
91_{16}	92_{17}	93_{2}	*94*	*95*	96_{3}	97_{17}	98_{6}	99_{28}	100_{4}	101_{6}	102_{1}	*103*	*104*	105_{1}	106_{7}	107_{6}	108_{5}
109_{14}	110_{1}	111_{5}	*112*	*113*	114_{0}	115_{6}	116_{6}	117_{4}	118_{23}	119_{9}	120_{9}	*121*	*122*	123_{2}	124_{4}	125_{∞}	126_{16}

(*complete up to* $|x|, |y|, |z| \leq 2 \cdot 10^7$)

Historical progression

Aug 1999: Bernstein implements Elkies method to find more solutions for $k \leq 1000$ searching up to $N = 3 \cdot 10^9$.

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{6}	7_{9}	8_{∞}	9_{7}	10_{6}	11_{8}	12_{2}	*13*	*14*	15_{6}	16_{∞}	17_{9}	18_{17}
19_{11}	20_{16}	21_{8}	*22*	*23*	24_{2}	25_{5}	26_{8}	27_{∞}	28_{14}	29_{18}	30_{1}	*31*	*32*	33_{0}	34_{20}	35_{18}	36_{15}
37_{5}	38_{6}	39_{1}	*40*	*41*	42_{0}	43_{17}	44_{7}	45_{12}	46_{9}	47_{10}	48_{9}	*49*	*50*	51_{4}	52_{0}	53_{13}	54_{∞}
55_{23}	56_{19}	57_{17}	*58*	*59*	60_{5}	61_{4}	62_{19}	63_{18}	64_{∞}	65_{8}	66_{3}	*67*	*68*	69_{7}	70_{17}	71_{25}	72_{8}
73_{13}	74_{0}	75_{1}	*76*	*77*	78_{2}	79_{5}	80_{11}	81_{9}	82_{7}	83_{28}	84_{3}	*85*	*86*	87_{2}	88_{14}	89_{8}	90_{34}
91_{17}	92_{22}	93_{3}	*94*	*95*	96_{3}	97_{23}	98_{7}	99_{32}	100_{7}	101_{8}	102_{1}	*103*	*104*	105_{2}	106_{8}	107_{6}	108_{6}
109_{18}	110_{1}	111_{6}	*112*	*113*	114_{0}	115_{7}	116_{10}	117_{5}	118_{27}	119_{11}	120_{13}	*121*	*122*	123_{2}	124_{4}	125_{∞}	126_{21}

(*complete up to* $|x|, |y|, |z| \leq 3 \cdot 10^9$)

Historical progression

Feb 2000: Beck, Pine, Tarrant, Yarbrough Jensen find first solution for $k = 52$,

$$(-61\,922\,712\,865)^3 + (23\,961\,292\,454)^3 + 60\,702\,901\,317^3 = 52.$$

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{6}	7_{9}	8_{∞}	9_{7}	10_{6}	11_{8}	12_{2}	*13*	*14*	15_{6}	16_{∞}	17_{9}	18_{17}
19_{11}	20_{16}	21_{8}	*22*	*23*	24_{2}	25_{5}	26_{8}	27_{∞}	28_{14}	29_{18}	30_{1}	*31*	*32*	33_{0}	34_{20}	35_{18}	36_{15}
37_{5}	38_{6}	39_{1}	*40*	*41*	42_{0}	43_{17}	44_{7}	45_{12}	46_{9}	47_{10}	48_{9}	*49*	*50*	51_{4}	52_{1}	53_{13}	54_{∞}
55_{23}	56_{19}	57_{17}	*58*	*59*	60_{5}	61_{4}	62_{19}	63_{18}	64_{∞}	65_{8}	66_{3}	*67*	*68*	69_{7}	70_{17}	71_{25}	72_{8}
73_{13}	74_{0}	75_{1}	*76*	*77*	78_{2}	79_{5}	80_{11}	81_{9}	82_{7}	83_{28}	84_{3}	*85*	*86*	87_{2}	88_{14}	89_{8}	90_{34}
91_{17}	92_{22}	93_{3}	*94*	*95*	96_{3}	97_{23}	98_{7}	99_{32}	100_{7}	101_{8}	102_{1}	*103*	*104*	105_{2}	106_{8}	107_{6}	108_{6}
109_{18}	110_{1}	111_{6}	*112*	*113*	114_{0}	115_{7}	116_{10}	117_{5}	118_{27}	119_{11}	120_{13}	*121*	*122*	123_{2}	124_{4}	125_{∞}	126_{21}

(*complete up to* $|x|, |y|, |z| \leq 3 \cdot 10^9$)

Historical progression

Feb 2008: Elsenhans, Jahnel (using Elkies method) compute all solutions for $k < 1000$ up to $|x|, |y|, |z| \leq 10^{14}$, finding a second solution for $k = 30$.

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{7}	7_{17}	8_{∞}	9_{13}	10_{10}	11_{12}	12_{2}	*13*	*14*	15_{10}	16_{∞}	17_{12}	18_{27}
19_{16}	20_{19}	21_{9}	*22*	*23*	24_{3}	25_{6}	26_{13}	27_{∞}	28_{16}	29_{19}	30_{2}	*31*	*32*	33_{0}	34_{28}	35_{26}	36_{24}
37_{8}	38_{11}	39_{1}	*40*	*41*	42_{0}	43_{25}	44_{15}	45_{23}	46_{14}	47_{15}	48_{12}	*49*	*50*	51_{7}	52_{2}	53_{16}	54_{∞}
55_{36}	56_{30}	57_{22}	*58*	*59*	60_{8}	61_{5}	62_{28}	63_{24}	64_{∞}	65_{10}	66_{4}	*67*	*68*	69_{8}	70_{29}	71_{34}	72_{17}
73_{25}	74_{0}	75_{3}	*76*	*77*	78_{3}	79_{8}	80_{17}	81_{16}	82_{13}	83_{48}	84_{4}	*85*	*86*	87_{2}	88_{20}	89_{13}	90_{45}
91_{22}	92_{32}	93_{3}	*94*	*95*	96_{5}	97_{34}	98_{15}	99_{53}	100_{8}	101_{8}	102_{1}	*103*	*104*	105_{3}	106_{18}	107_{9}	108_{7}
109_{22}	110_{2}	111_{7}	*112*	*113*	114_{0}	115_{9}	116_{15}	117_{12}	118_{35}	119_{17}	120_{17}	*121*	*122*	123_{3}	124_{8}	125_{∞}	126_{31}

(*complete up to* $|x|, |y|, |z| \leq 10^{14}$)

Historical progression

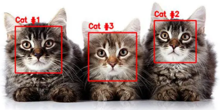
Dec 2011:

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{7}	7_{17}	8_{∞}	9_{13}	10_{10}	11_{12}	12_{2}	*13*	*14*	15_{10}	16_{∞}	17_{12}	18_{27}
19_{16}	20_{19}	21_{9}	*22*	*23*	24_{3}	25_{6}	26_{13}	27_{∞}	28_{16}	29_{19}	30_{2}	*31*	*32*	33_{0}	34_{28}	35_{26}	36_{24}
37_{8}	38_{11}	39_{1}	*40*	*41*	42_{0}	43_{25}	44_{15}	45_{23}	46_{14}	47_{15}	48_{12}	*49*	*50*	51_{7}	52_{2}	53_{16}	54_{∞}
55_{36}	56_{30}	57_{22}	*58*	*59*	60_{8}	61_{5}	62_{28}	63_{24}	64_{∞}	65_{10}	66_{4}	*67*	*68*	69_{8}	70_{29}	71_{34}	72_{17}
73_{25}	74_{0}	75_{3}	*76*	*77*	78_{3}	79_{8}	80_{17}	81_{16}	82_{13}	83_{48}	84_{4}	*85*	*86*	87_{2}	88_{20}	89_{13}	90_{45}
91_{22}	92_{32}	93_{3}	*94*	*95*	96_{5}	97_{34}	98_{15}	99_{53}	100_{8}	101_{8}	102_{1}	*103*	*104*	105_{3}	106_{18}	107_{9}	108_{7}
109_{22}	110_{2}	111_{7}	*112*	*113*	114_{0}	115_{9}	116_{15}	117_{12}	118_{35}	119_{17}	120_{17}	*121*	*122*	123_{3}	124_{8}	125_{∞}	126_{31}

(*complete up to* $|x|, |y|, |z| \leq 10^{14}$)

Historical progression

Dec 2011: computers learn to identify cats



1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{7}	7_{17}	8_{∞}	9_{13}	10_{10}	11_{12}	12_{2}	*13*	*14*	15_{10}	16_{∞}	17_{12}	18_{27}
19_{16}	20_{19}	21_{9}	*22*	*23*	24_{3}	25_{6}	26_{13}	27_{∞}	28_{16}	29_{19}	30_{2}	*31*	*32*	33_{0}	34_{28}	35_{26}	36_{24}
37_{8}	38_{11}	39_{1}	*40*	*41*	42_{0}	43_{25}	44_{15}	45_{23}	46_{14}	47_{15}	48_{12}	*49*	*50*	51_{7}	52_{2}	53_{16}	54_{∞}
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73_{25}	74_{0}	75_{3}	*76*	*77*	78_{3}	79_{8}	80_{17}	81_{16}	82_{13}	83_{48}	84_{4}	*85*	*86*	87_{2}	88_{20}	89_{13}	90_{45}
91_{22}	92_{32}	93_{3}	*94*	*95*	96_{5}	97_{34}	98_{15}	99_{53}	100_{8}	101_{8}	102_{1}	*103*	*104*	105_{3}	106_{18}	107_{9}	108_{7}
109_{22}	110_{2}	111_{7}	*112*	*113*	114_{0}	115_{9}	116_{15}	117_{12}	118_{35}	119_{17}	120_{17}	*121*	*122*	123_{3}	124_{8}	125_{∞}	126_{31}

(*complete up to* $|x|, |y|, |z| \leq 10^{14}$)

Historical progression

Nov 2015: Brady Haran releases a Numberphile video featuring Tim Browning, titled "The Uncracked Problem with 33".

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{7}	7_{17}	8_{∞}	9_{13}	10_{10}	11_{12}	12_{2}	*13*	*14*	15_{10}	16_{∞}	17_{12}	18_{27}
19_{16}	20_{19}	21_{9}	*22*	*23*	24_{3}	25_{6}	26_{13}	27_{∞}	28_{16}	29_{19}	30_{2}	*31*	*32*	33_{0}	34_{28}	35_{26}	36_{24}
37_{8}	38_{11}	39_{1}	*40*	*41*	42_{0}	43_{25}	44_{15}	45_{23}	46_{14}	47_{15}	48_{12}	*49*	*50*	51_{7}	52_{2}	53_{16}	54_{∞}
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91_{22}	92_{32}	93_{3}	*94*	*95*	96_{5}	97_{34}	98_{15}	99_{53}	100_{8}	101_{8}	102_{1}	*103*	*104*	105_{3}	106_{18}	107_{9}	108_{7}
109_{22}	110_{2}	111_{7}	*112*	*113*	114_{0}	115_{9}	116_{15}	117_{12}	118_{35}	119_{17}	120_{17}	*121*	*122*	123_{3}	124_{8}	125_{∞}	126_{31}

(*complete up to* $|x|, |y|, |z| \leq 10^{14}$)

Historical progression

Apr 2016: Huisman searches up to $N = 10^{15}$. Finds first solution for $k = 74$,

$$(-284\,650\,292\,555\,885)^3 + 66\,229\,832\,190\,556^3 + 283\,450\,105\,697\,727^3 = 74.$$

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{8}	7_{18}	8_{∞}	9_{15}	10_{10}	11_{12}	12_{2}	*13*	*14*	15_{10}	16_{∞}	17_{13}	18_{27}
19_{16}	20_{21}	21_{9}	*22*	*23*	24_{3}	25_{6}	26_{13}	27_{∞}	28_{19}	29_{21}	30_{3}	*31*	*32*	33_{0}	34_{28}	35_{27}	36_{26}
37_{8}	38_{11}	39_{1}	*40*	*41*	42_{0}	43_{25}	44_{16}	45_{26}	46_{16}	47_{16}	48_{12}	*49*	*50*	51_{7}	52_{2}	53_{18}	54_{∞}
55_{42}	56_{31}	57_{23}	*58*	*59*	60_{8}	61_{5}	62_{33}	63_{26}	64_{∞}	65_{11}	66_{4}	*67*	*68*	69_{10}	70_{31}	71_{34}	72_{17}
73_{27}	74_{1}	75_{4}	*76*	*77*	78_{3}	79_{9}	80_{19}	81_{19}	82_{13}	83_{49}	84_{4}	*85*	*86*	87_{3}	88_{21}	89_{17}	90_{48}
91_{23}	92_{33}	93_{3}	*94*	*95*	96_{5}	97_{37}	98_{18}	99_{56}	100_{9}	101_{10}	102_{1}	*103*	*104*	105_{3}	106_{18}	107_{10}	108_{7}
109_{25}	110_{2}	111_{8}	*112*	*113*	114_{0}	115_{10}	116_{15}	117_{12}	118_{36}	119_{18}	120_{17}	*121*	*122*	123_{3}	124_{8}	125_{∞}	126_{33}

(*complete up to* $|x|, |y|, |z| \leq 10^{15}$)

Historical progression

Feb 2019: Booker finds first solution for $k = 33$,

$$(-8\,778\,405\,442\,862\,239)^3 + (-2\,736\,111\,468\,807\,040)^3 + (8\,866\,128\,975\,287\,528)^3 = 33.$$

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{8}	7_{18}	8_{∞}	9_{15}	10_{10}	11_{12}	12_{2}	*13*	*14*	15_{10}	16_{∞}	17_{13}	18_{27}
19_{16}	20_{21}	21_{9}	*22*	*23*	24_{3}	25_{6}	26_{13}	27_{∞}	28_{19}	29_{21}	30_{3}	*31*	*32*	33_{1}	34_{28}	35_{27}	36_{26}
37_{8}	38_{11}	39_{1}	*40*	*41*	42_{0}	43_{25}	44_{16}	45_{26}	46_{16}	47_{16}	48_{12}	*49*	*50*	51_{7}	52_{2}	53_{18}	54_{∞}
55_{42}	56_{31}	57_{23}	*58*	*59*	60_{8}	61_{5}	62_{33}	63_{26}	64_{∞}	65_{11}	66_{4}	*67*	*68*	69_{10}	70_{31}	71_{34}	72_{17}
73_{27}	74_{1}	75_{4}	*76*	*77*	78_{3}	79_{9}	80_{19}	81_{19}	82_{13}	83_{49}	84_{4}	*85*	*86*	87_{3}	88_{21}	89_{17}	90_{48}
91_{23}	92_{33}	93_{3}	*94*	*95*	96_{5}	97_{37}	98_{18}	99_{56}	100_{9}	101_{10}	102_{1}	*103*	*104*	105_{3}	106_{18}	107_{10}	108_{7}
109_{25}	110_{2}	111_{8}	*112*	*113*	114_{0}	115_{10}	116_{15}	117_{12}	118_{36}	119_{18}	120_{17}	*121*	*122*	123_{3}	124_{8}	125_{∞}	126_{33}

(*complete up to* $|x|, |y|, |z| \leq 10^{15}$)

Historical progression

Sep 2019: Booker, Sutherland find first solution for $k = 42$,

$$(-80\,538\,738\,812\,075\,974)^3 + 80\,435\,758\,145\,817\,515^3 + 12\,602\,123\,297\,335\,631^3 = 42.$$

1_{∞}	2_{∞}	3_{2}	*4*	*5*	6_{8}	7_{18}	8_{∞}	9_{15}	10_{10}	11_{12}	12_{2}	*13*	*14*	15_{10}	16_{∞}	17_{13}	18_{27}
19_{16}	20_{21}	21_{9}	*22*	*23*	24_{3}	25_{6}	26_{13}	27_{∞}	28_{19}	29_{21}	30_{3}	*31*	*32*	33_{1}	34_{28}	35_{27}	36_{26}
37_{8}	38_{11}	39_{1}	*40*	*41*	42_{1}	43_{25}	44_{16}	45_{26}	46_{16}	47_{16}	48_{12}	*49*	*50*	51_{7}	52_{2}	53_{18}	54_{∞}
55_{42}	56_{31}	57_{23}	*58*	*59*	60_{8}	61_{5}	62_{33}	63_{26}	64_{∞}	65_{11}	66_{4}	*67*	*68*	69_{10}	70_{31}	71_{34}	72_{17}
73_{27}	74_{1}	75_{4}	*76*	*77*	78_{3}	79_{9}	80_{19}	81_{19}	82_{13}	83_{49}	84_{4}	*85*	*86*	87_{3}	88_{21}	89_{17}	90_{48}
91_{23}	92_{33}	93_{3}	*94*	*95*	96_{5}	97_{37}	98_{18}	99_{56}	100_{9}	101_{10}	102_{1}	*103*	*104*	105_{3}	106_{18}	107_{10}	108_{7}
109_{25}	110_{2}	111_{8}	*112*	*113*	114_{0}	115_{10}	116_{15}	117_{12}	118_{36}	119_{18}	120_{17}	*121*	*122*	123_{3}	124_{8}	125_{∞}	126_{33}

(*complete up to* $|x|, |y|, |z| \leq 10^{15}$)

Historical progression

Sep 2019: Booker, Sutherland find third solution for $k = 3$,

$$569936821221962380720^3 + (-569936821113563493509)^3 + (-472715493453327032)^3 = 3.$$

1_{∞}	2_{∞}	3_{3}	*4*	*5*	6_{8}	7_{18}	8_{∞}	9_{15}	10_{10}	11_{12}	12_{2}	*13*	*14*	15_{10}	16_{∞}	17_{13}	18_{27}
19_{16}	20_{21}	21_{9}	*22*	*23*	24_{3}	25_{6}	26_{13}	27_{∞}	28_{19}	29_{21}	30_{3}	*31*	*32*	33_{1}	34_{28}	35_{27}	36_{26}
37_{8}	38_{11}	39_{1}	*40*	*41*	42_{1}	43_{25}	44_{16}	45_{26}	46_{16}	47_{16}	48_{12}	*49*	*50*	51_{7}	52_{2}	53_{18}	54_{∞}
55_{42}	56_{31}	57_{23}	*58*	*59*	60_{8}	61_{5}	62_{33}	63_{26}	64_{∞}	65_{11}	66_{4}	*67*	*68*	69_{10}	70_{31}	71_{34}	72_{17}
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(*complete up to* $|x|, |y|, |z| \leq 10^{15}$)

Algorithms

- Naive brute force algorithm:

for $x = -N$ to N **do**
    for $y = -N$ to N **do**
        for $z = -N$ to N **do**
            if $x^3 + y^3 + z^3 = k$ **then**
                return (x, y, z)

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- Slightly less naive algorithm:

---

for $x = -N$ to N **do**
  for $y = -N$ to N **do**
    if $\sqrt[3]{k - x^3 - y^3} \in \mathbb{Z}$ **then**
      return $(x, y, \sqrt[3]{k - x^3 - y^3})$

Algorithms

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Runtime: $O(N^2 \log N)$

Algorithms

- Use sum of two cubes algorithm.

---

for $x = -N$ to N **do**
  for all divisors d of $k - x^3$ **do** ▷ Checking if $k - x^3$ is sum of two cubes
    if $\sqrt{\frac{4(k-x^3)-d^3}{3d}} \in \mathbb{Z}$ **then**
      return $\left(x,\ \frac{d}{2} + \frac{1}{2}\sqrt{\frac{4(k-x^3)-d^3}{3d}},\ \frac{d}{2} - \frac{1}{2}\sqrt{\frac{4(k-x^3)-d^3}{3d}}\right)$

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Runtime: $O(N^{1+o(1)})$ (depends on time complexity of integer factorisation)

e.g. using GNFS, $O\big(N \exp((c(\log N)^{1/3}(\log\log N)^{2/3})\big)$ for some small constant c.

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e.g. using GNFS, $O\big(N \exp((c(\log N)^{1/3}(\log\log N)^{2/3})\big)$ for some small constant c.

This obtains all solutions where $\min(|x|, |y|, |z|) \leq N$.

Algorithms

- Let's first search through d, then filter the values of x.

---

for all $d = -2N$ to $2N$ **do**
  for all $x \in [-N, N]$ such that $x^3 \equiv k \pmod{d}$ **do**
    if $\sqrt{\frac{4(k-x^3)-d^3}{3d}} \in \mathbb{Z}$ **then**
      return $\left(x,\ \frac{d}{2} + \frac{1}{2}\sqrt{\frac{4(k-x^3)-d^3}{3d}},\ \frac{d}{2} - \frac{1}{2}\sqrt{\frac{4(k-x^3)-d^3}{3d}}\right)$

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Solving $x^3 \equiv k \pmod d$ can be done by factoring $x^3 - k$ over $\mathbb{F}_p$ for all primes $p|d$, and using Hensel lifting and the Chinese remainder theorem.

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Runtime: $O(N^{1+o(1)})$

(here we're factorising numbers of size $O(N)$ instead of $O(N^3)$).

Algorithms

Some further optimisations:

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- We may assume wlog $|x| \leq |y| \leq |z|$. If we also assume $x \neq y$ and $|x| > \sqrt{k}$, then a simple argument proves that $|y + z| < \alpha|x|$ where $\alpha = \sqrt[3]{2} - 1 \approx 0.26$. Thus we may assume $|d| < \alpha N$.

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- For particular values of k, one can often derive additional congruential restrictions on x, y, z. E.g. if $k = 3$, then cubic reciprocity implies $x \equiv y \equiv z \pmod 9$.

Algorithms

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Solve the Thue equation $x^3 + 2y^3 = k$.

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for all $d \in [-\alpha N, \alpha N]$, ordered by divisibility **do**
    for all $x \in [-N, N]$ such that $x^3 \equiv k \pmod{d}$ **do**
        if $\sqrt{\frac{4(k-x^3)-d^3}{3d}} \in \mathbb{Z}$ **then**
            return $\left(x,\ \frac{d}{2} + \frac{1}{2}\sqrt{\frac{4(k-x^3)-d^3}{3d}},\ \frac{d}{2} - \frac{1}{2}\sqrt{\frac{4(k-x^3)-d^3}{3d}}\right)$

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Runtime: $O(N \log^{O(1)}(N))$

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Solve the Thue equation $x^3 + 2y^3 = k$.
Naively brute force solutions for $|x| \leq \sqrt{k}$.
for all primes $p < \alpha N$ **do**
    precompute solutions to $x^3 \equiv k \pmod{p}$
for all $d \in [-\alpha N, \alpha N]$, ordered by divisibility **do**
    for all $x \in [-N, N]$ such that $x^3 \equiv k \pmod{d}$ **do**
        if $\sqrt{\frac{4(k-x^3)-d^3}{3d}} \in \mathbb{Z}$ **then**
            return $\left(x,\ \frac{d}{2} + \frac{1}{2}\sqrt{\frac{4(k-x^3)-d^3}{3d}},\ \frac{d}{2} - \frac{1}{2}\sqrt{\frac{4(k-x^3)-d^3}{3d}}\right)$

Runtime: $O(N \log^{O(1)}(N))$
By applying many more optimisations, and filtering down the pairs (x, d) to consider, Booker gets an algorithm that's (almost) linear in practice: $O(N \log\log N \log\log\log N)$.

Thank you!