$$E\langle \xi, \varphi\rangle\langle \xi, \psi\rangle = \langle \varphi, \Psi\rangle$$

$$E\int \xi(x)\cdot\varphi(x)\,dx \int \xi(y)\cdot\psi(y)\,dy$$

$$= \int \varphi(x)\cdot\Big(\underbrace{\int \psi(y)\cdot E\xi(x)\xi(y)\,dy}_{\psi(x)}\Big)\,dx$$

$$\Rightarrow \underline{E\xi(x)\xi(y) = \delta(x-y)}$$

$\underline{\partial_t u = \Delta u + \xi}$, u_0

$u: \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$

ξ is space-time white noise

$$E\,\xi(s,x)\,\xi(t,y) = \delta(s-t)\cdot\delta(x-y)$$

$$u(t,x) = \underline{\int_0^t (H_{t-s} * \xi)(x)\,ds} + \underbrace{H_t * u_0}_{0}$$

$$H_t(x) = (4\pi t)^{-\frac{d}{2}}\cdot e^{-\frac{|x|^2}{4t}}$$

$$E\,|u(s,x) - u(t,y)|^2$$

$$= E\,|u(s,x)|^2 + E|u(t,y)|^2 - 2\underline{E u(s,x)\,u(t,y)}$$

$$C(s,t,x,y) = E\,u(s,x)\,u(t,y)$$

$$= E\,u(s,0)\,u(t,y-x)$$

need to compute. $C(s,t,0,x)$

$$C(s,t,0,x) = E\,u(s,0)\,u(t,x)$$

$$= E\int_0^s\Big(\int_{\mathbb{R}^d} H_{s-r}(-y)\cdot\xi(r,y)\,dy\Big)dr$$
$$\int_0^t\Big(\int_{\mathbb{R}^d} H_{t-r'}(x-y')\cdot\xi(r',y')\,dy'\Big)dr'$$

$$= \int_0^s\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H_{s-r}(y)\cdot H_{t-r'}(x-y')$$
$$\cdot\,\delta(r-r')\cdot\delta(y-y')\,dy\,dy'\,dr\,dr'$$

$$= \int_0^{s\wedge t}\underbrace{\int_{\mathbb{R}^d} H_{s-r}(y)\cdot H_{t-r}(x-y)\,dy}_{\substack{(H_{s-r} * H_{t-r})(x)\\ = H_{s+t-2r}(x)}}\,dr$$

$$= \int_0^{s\wedge t} H_{s+t-2r}(x)\,dr$$

$$= (4\pi)^{-\frac{d}{2}}\cdot\int_0^{s\wedge t}(\underbrace{s+t-2r}_{=\ell})^{-\frac{d}{2}}\cdot e^{-\frac{|x|^2}{4(s+t-2r)}}\,dr$$

$$= \underline{C\cdot\int_{|t-s|}^{t+s}\ell^{-\frac{d}{2}}\cdot e^{-\frac{|x|^2}{4\ell}}\,d\ell}$$

$$= E\,u(s,0)\,u(t,x)$$

$$E\, u(s,0)\, u(t,x)$$

$$\sim \int_{t-s}^{t+s} \ell^{-\frac{d}{2}} \cdot e^{-\frac{|x|^2}{4\ell}} \, d\ell$$

- $s=t,\ x=0$

$$\Rightarrow E|u(t,0)|^2 = E|u(t,x)|^2$$

$d=1$:

$$C(s,t,0,x) \sim \int_{t-s}^{t+s} \ell^{-\frac{1}{2}} \cdot e^{-\frac{|x|^2}{4\ell}} \, d\ell$$

(1) For fixed space pt., compute time regularity

$$\Rightarrow C(s,t,0,0)$$

$$= \int_{t-s}^{t+s} \ell^{-\frac{1}{2}} d\ell$$

$$= \cancel{2} \left((t+s)^{\frac{1}{2}} - (\underbrace{t-s}_{h})^{\frac{1}{2}} \right)$$

$$E|u(s,0) - u(t,0)|^2$$

$$= E|u(s,0)|^2 + E|u(t,0)|^2 - 2E\, u(s,0) u(t,0)$$

$$\sim (2s)^{\frac{1}{2}} + (2s+2h)^{\frac{1}{2}} - 2\left[(2s+h)^{\frac{1}{2}} - h^{\frac{1}{2}}\right]$$

$$= \underbrace{(2s)^{\frac{1}{2}} - (2s+h)^{\frac{1}{2}}}_{O(h^{\frac{1}{2}})} + \underbrace{(2s+2h)^{\frac{1}{2}} - (2s+h)^{\frac{1}{2}}}_{O(h^{\frac{1}{2}})} + h^{\frac{1}{2}}$$

$$< C \cdot h^{\frac{1}{2}}$$

$$E|u(s,x) - u(t,y)|^2 < C \cdot |t-s|^{\frac{1}{2}}$$

$$\Rightarrow |u(s,x) - u(t,x)| < C \cdot |t-s|^{\frac{1}{4}-k}$$

a.s. for any $k>0$

$$\Rightarrow \boxed{u(\cdot, x) \text{ is almost Hölder-}\tfrac{1}{4}}$$

(2) Fix time t, compute spatial regularity

$$C(t,t,0,x) \sim \int_0^{2t} \ell^{-\frac{1}{2}} \cdot e^{-\frac{|x|^2}{4\ell}} \, d\ell$$

$$z = \frac{|x|^2}{4\ell}$$

$$\Rightarrow C(t,t,0,x)$$

$$= \int_{+\infty}^{\frac{|x|^2}{8t}} \left(\frac{4z}{|x|^2}\right)^{\frac{1}{2}} \cdot e^{-z} \cdot d\frac{|x|^2}{4z}$$

$$= \frac{|x|}{2} \cdot \int_{\frac{|x|^2}{8t}}^{+\infty} z^{-\frac{3}{2}} \cdot e^{-z} \, dz$$

$$=$$

$$\sim \frac{|x|}{2} \cdot \int_{\frac{|x|^2}{8t}}^{+\infty} z^{-\frac{3}{2}} \cdot e^{-z}\, dz$$

$$= \frac{|x|}{\cancel{2}} \cdot \cancel{2} \int_{\frac{|x|^2}{8t}}^{+\infty} e^{-z}\, dz^{-\frac{1}{2}}$$

$$= -|x| \cdot \left[z^{-\frac{1}{2}} e^{-z} \Big|_{\frac{|x|^2}{8t}}^{+\infty} - \int_{\frac{|x|^2}{8t}}^{+\infty} z^{-\frac{1}{2}} d e^{-z} \right]$$

$$= \underbrace{\sqrt{8t} \cdot e^{-\frac{|x|^2}{8t}}}_{\Downarrow} - |x| \cdot \underbrace{\int_{\frac{|x|^2}{8t}}^{+\infty} z^{-\frac{1}{2}} e^{-z} dz}_{\downarrow}$$

$$\sqrt{8t}\left(1 - \frac{|x|^2}{8t} + O(|x|^4)\right) \qquad O(|x|)$$

$$= \underbrace{\sqrt{8t} + C|x| + O(|x|^2)}_{\Rightarrow E\, u(t,0)\, u(t,x)}$$

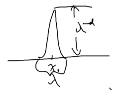
$$\Rightarrow E|u(t,0) - u(t,x)|^2 < C \cdot |x|$$

$\Rightarrow u(t,\cdot)$ is almost Hölder-$\frac{1}{2}$.

$f \in C^\alpha$, $\alpha \in (0,1)$

$$|f(x) - f(y)| \lesssim |x-y|^\alpha$$

$$\varphi_{x_0}^\lambda(x) = \lambda^{-d} \varphi\left(\frac{x - x_0}{\lambda}\right)$$



$$\langle f - f(x_0), \varphi_{x_0}^\lambda \rangle \lesssim \lambda^\alpha$$

$f \in C^{-\alpha}$ if

$$\langle f, \varphi_x^\lambda \rangle \lesssim \lambda^{-\alpha}$$

$$\langle \delta, \varphi_x^\lambda \rangle$$

$$= \lambda^{-d} \int \delta(y) \cdot \varphi\left(\frac{y-x}{\lambda}\right) \cdot dy$$

$$= \lambda^{-d} \cdot \varphi\left(-\frac{x}{\lambda}\right) \leq C \cdot \lambda^{-d}$$

$$\partial_t u = \Delta u + \xi$$

$$u(t,x) = \int_0^t (H_{t-s} * \xi(s,\cdot))(x)\, ds$$

$$= (H \underset{\downarrow}{*} \xi)(t,x)$$

space-time convolution

$$H(t,x) = (2\pi t)^{-\frac{d}{2}} \cdot e^{-\frac{|x|^2}{4t}}$$

$$H(\lambda^2 t, \lambda x) = \lambda^{-d} \cdot H(t,x)$$

$$z = (t,x)$$

$$|z-z'| = |t-t'|^{\frac{1}{2}} + |x-x'|$$

$$H(\underbrace{\lambda^2 t, \lambda x}_{\lambda z}) = \lambda^{-d} \cdot H(\underbrace{t, x}_{z})$$

For small z, $H(z) \sim |z|^{-d}$

$K(x) \sim |x|^{-d+\beta}$. $f \in C^{\alpha}$

$K * f \in C^{\alpha+\beta}$

$H(z) \sim |z|^{-(d+2)+2}$

$u = H * \xi$

$\xi \in C^{-\frac{d}{2}-1-}$

$\Rightarrow u \in C^{1-\frac{d}{2}-}$

$H = K + \overline{K}$

$u = H * \xi$

$\Rightarrow u = \boxed{K * \xi} + \underbrace{\overline{K} * \xi}_{\checkmark}$

$<$

$$\langle K * \xi, \varphi \rangle$$

$$= \int \left(\int k(x) \cdot \xi(y-x)\, dx \right) \varphi(y)\, dy$$

$$= \int k(x) \cdot \left(\int \xi(y) \cdot \varphi(y+x)\, dy \right) dx$$

$$= \int k(x) \cdot \langle \xi, \varphi_x \rangle\, dx$$

$$v = k * \xi$$

$$E|\langle v, \varphi \rangle|^2$$

$$= E \int k(x) \langle \xi, \varphi_x \rangle dx \cdot \int k(y) \langle \xi, \varphi_y \rangle dy$$

$$= \iint k(x) k(y) \cdot \langle \varphi_x, \varphi_y \rangle\, dx\, dy$$

$$= \iint k(x) k(y) \left(\int \varphi(x+z) \cdot \varphi(y+z)\, dz \right) dx\, dy$$

$$= \int \left(\int k(x-z) \cdot \varphi(x)\, dx \right)^2 dz$$

$$\Rightarrow E|\langle v, \varphi^\lambda \rangle|^2$$

$$= \int_{|z| \le 1} \left(\int_{|x| \le \lambda} k(x-z) \cdot \varphi^\lambda(x)\, dx \right) dz$$

$$= \int_{|z| \le 1} \left(\int k\left(\lambda \frac{x-z}{\lambda}\right) \varphi\left(\frac{x}{\lambda}\right) d\frac{x}{\lambda} \right)^2 dz$$

$$= \lambda^{2-d} \int_{|z| \le 1} \left(\int k\left(x - \frac{z}{\lambda}\right) \cdot \varphi(x)\, dx \right)^2 d\frac{z}{\lambda}$$

$$= \lambda^{2-d} \int_{|z| \le \frac{1}{\lambda}} \underbrace{\left(\int \tilde{k}(x-z) \cdot \varphi(x)\, dx \right)^2}_{F(z) \sim |z|^{-2d}} dz$$

$$\tilde{k}(z) \sim |z|^{-d} \text{ for large } z$$

$$\sim \lambda^{2-d} \cdot \int_{|z| \le \frac{1}{\lambda}} |z|^{-2d}\, dz$$

$$\sim \begin{cases} |\log \lambda|, & d = 2 \\ \lambda^{2-d}, & d \ge 3 \end{cases}$$

$$E|\langle u, \varphi^\lambda \rangle|^2 \lesssim \begin{cases} |\log \lambda|, & d = 2 \\ \lambda^{2-d}, & d \ge 3 \end{cases}$$

$$\Rightarrow u \in C^{1 - \frac{d}{2} -}$$

$$\partial_t u = \Delta u + \xi$$

$$\mathcal{L}$$

$$d u_t = \Delta u_t \cancel{+ \xi}$$

$$u_t = e^{\Delta t} u_0 \longrightarrow e^{\Delta t} u_0$$