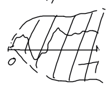
$\mu(dx)$ on X

$\mu(A) = Pr(x \in A)$

$x \longmapsto x+h, \quad h \in X$

$\mu^h(A) := Pr(x+h \in A)$

$\frac{d\mu^h}{d\mu} \stackrel{?}{=} f$



$A = A_1 \times A_2 \times \cdots \times A_n$

$\mu(A) \simeq Pr(X_{t_1} \in A_1, \cdots, X_{t_n} \in A_n)$

$$= \int \cdots \int_{A_1 \times \cdots \times A_n} \prod_{j=0}^{n-1} p_{t_{j+1}-t_j}(X_{t_{j+1}} - X_{t_j}) \, dx_{t_1} \cdots dx_{t_n}$$

$$= \int \cdots \int_{A_1 \times \cdots \times A_n} \prod_j (2\pi(t_{j+1}-t_j))^{-\frac{1}{2}} \cdot e^{-\frac{|X_{t_{j+1}} - X_{t_j}|^2}{2(t_{j+1}-t_j)}} \, dx_{t_j}$$

$\mu^h(A) = Pr(x+h \in A)$

$$= \int \cdots \int_A \prod_j ((2\pi(t_{j+1}-t_j))^{-\frac{1}{2}} \cdot e^{-\frac{|x_{t_{j+1}}+h_{t_{j+1}} - x_{t_j} - h_{t_j}|^2}{2(t_{j+1}-t_j)}} \, dx_{t_j}$$

$$= \int \cdots \int_A \prod_j (2\pi \Delta t_j)^{-\frac{1}{2}} \cdot \boxed{\exp\left[-\sum_j \frac{|\Delta x_{t_j} + \Delta h_{t_j}|^2}{2\, \Delta t_j}\right]} \prod dx_{t_j}$$

$$\exp(\cdots) \doteq -\sum_j \frac{1}{2\Delta t_j} \left(\underline{\Delta x_{t_j}^2} + \underline{2\Delta x_{t_j} \Delta h_{t_j}} + \underline{\Delta h_{t_j}^2}\right)$$

$$\sum_j \Delta x_{t_j} \cdot \frac{\Delta h_{t_j}}{\Delta t_j} \sim \int_0^1 h'(t) \, dx(t)$$

$$\frac{1}{2} \sum_j \frac{\Delta h_{t_j}^2}{\Delta t_j^2} \cdot \Delta t_j \sim \frac{1}{2} \int_0^1 (h'(t))^2 \, dt$$

$\mu^h(A)$

$$\sim \int \cdots \int_A \prod_j (2\pi \Delta t_j)^{-\frac{1}{2}} \cdot e^{-\frac{\Delta x_{t_j}^2}{\Delta t_j}} \boxed{e^{-\int_0^1 h'(t) dx(t)} \cdot e^{-\frac{1}{2}\int_0^1 h'(t)^2 dt}} \, dx_{t_1} \cdots dx_{t_n}$$

$$\mu^h(dx) = \underbrace{\underline{e^{-\int_0^1 h'(t) dx(t) - \frac{1}{2}\int_0^1 h'(t)^2 dt}}}_{f_h(x)} \mu(dx)$$

In order for $\underline{f_h}$ to $\underline{\text{exist}}$,
we need h to be a.c.
and $\int_0^1 |h'(t)|^2 dt < +\infty$

$x \in \mathbb{R}^d$

$$\mu(dx) \sim \exp\left(-\tfrac{1}{2}\langle Q^{-1}x, x\rangle\right) dx$$

where Q is the covariance matrix.

$$\mathbb{E}\langle x, f\rangle\langle x, g\rangle = \langle Qf, g\rangle$$

In the case of infinite dimensional space, $(x \in X)$. take $f, g \in X^*$

$$\mathbb{E} f(x) g(x) = \langle Qf, g\rangle$$

B ~ Brownian bridge on $[0,1]$

$$\mathbb{E} B_s B_t = s \wedge t - st$$

$$\langle Qf, g\rangle = \mathbb{E}\langle B, f\rangle\langle B, g\rangle$$

$$= \mathbb{E}\int_0^1 B(s) f(s)\, ds \int_0^1 B(t) g(t)\, dt$$

$$= \iint_{[0,1]^2} (s \wedge t - st)\cdot f(s) g(t)\, ds\, dt$$

$$= \int_0^1 \Bigg(\underbrace{\int_0^1 (s\wedge t - st) f(s)\, ds}_{(Qf)(t)}\Bigg) g(t)\, dt$$

$$(Qf)(t) = \int_0^t s(1-t) f(s)\, ds + \int_t^1 t(1-s) f(s)\, ds$$

$$= \int_0^t s f(s)\, ds + t\int_t^1 f(s)\, ds - t\cdot\int_0^1 s f(s)\, ds$$

$$(Qf)'(t) = \cancel{t f(t)} + \int_t^1 f(s)\, ds \cancel{- t f(t)} - \int_0^1 s f(s)\, ds$$

$$(Qf)''(t) = -f(t)$$

$*$ $Q = (-\Delta)^{-1}$

$(Qf)(0) = (Qf)(1) = 0$