Wachspress Coordinates
University of Warwick

WACHSPRESS COORDINATES
A BRIDGE BETWEEN ALGEBRA, GEOMETRY AND COMBINATORICS

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THE WACHSPRESS FAMILY

Iz mestiev matrix

Wachspress

coordinates
variety
ideal
map
point

spectral
center
coordinates

adjoint polynomial
adjoint hypersurface

positive geometries
Barycentric coordinates

\[ x = \sum_{i} \alpha_i(x) p_i, \quad \alpha \in \Delta_n := \{\alpha \in [0, 1]^n \mid \alpha_1 + \cdots + \alpha_n = 1\}. \]
Barycentric coordinates for polytopes (??)

\[ x = \sum_i \alpha_i(x) p_i, \quad \alpha \in \Delta_n := \{ \alpha \in [0, 1]^n \mid \alpha_1 + \cdots + \alpha_n = 1 \}. \]
Application: interpolation

\[ \text{DATA}(x) = \sum_i \alpha_i(x) \text{DATA}(p_i) \]

▶ computer graphics
▶ finite element analysis
▶ ...

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APPLICATION: IMAGE WARPING
Generalized barycentric coordinates (GBCs): \( \alpha : P \to \Delta_n \) satisfy

\[
\sum_i \alpha_i(x)p_i = x \quad \text{(linear precision)}
\]
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$$\sum_i \alpha_i(x)p_i = x \quad \text{linear precision}$$

There are ...

- harmonic coordinates,
- mean value coordinates,
- ...
- Wachspress coordinates

(Wachspress, 1975; Warren, 1996)
The many faces of Wachspress coordinates
The many faces of Wachspress coordinates

WACHSPRESS COORDINATES AS RATIONAL GBCs

- There do not always exist polynomial GBCs. (Wachspress)
- Wachspress constructed rational GBCs:

\[ \alpha_i(x) = \frac{p_i(x)}{q(x)} \]

where \( q(x) = \sum_i p_i(x) \) ... adjoint polynomial
The many faces of Wachspress coordinates

**Wachspress coordinates as rational GBCs**

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where \(q(x) = \sum_i p_i(x)\) ... adjoint polynomial

**Idea:** if \(x \in \text{face}_k\) but \(p_i \not\in \text{face}_k\), then \(\alpha_i(x) = 0\):

\[
p_i(x) = \beta_i(x) \prod_{k: i \not\in \text{face}_k} H_k(x).
\]
The many faces of Wachspress coordinates

**Wachspress coordinates as rational GBCs**

- There do not always exist *polynomial* GBCs. 
- Wachspress constructed *rational* GBCs:

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where \( q(x) = \sum_i p_i(x) \ldots \) adjoint polynomial

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**Theorem.** *(Warren)*

The Wachspress coordinates are the unique rational GBCs of lowest possible degree.  
\[ \text{degree} = \#\text{facets} - \text{dim} \]
WACHSPRESS IN ALGEBRAIC GEOMETRY

- **Wachspress variety** \( V(P) := \text{im}(\alpha) \subseteq \Delta_n \)
- **Wachspress ideal** \( I(P) \)
- **adjoint hypersurface** ... vanishing set of adjoint polynomial
polar dual ... \( P^\circ := \{ x \in \mathbb{R}^d \mid \langle x, p_i \rangle \leq 1 \text{ for all } i \in V(G_P) \} \).

\[ \alpha_i := \frac{\text{vol}(F_i^\circ)}{\|p_i\| \text{vol}(P^\circ)} \]
The many faces of Wachspress coordinates

WACHSPRESS FROM CONE VOLUMES (Ju et al., 2005)

polar dual ... \( P^\circ := \{ x \in \mathbb{R}^d \mid \langle x, p_i \rangle \leq 1 \text{ for all } i \in V(G_P) \} \).

\[ \alpha_i := \frac{\text{vol}(F_i^\circ)}{\| p_i \| \cdot \text{vol}(P^\circ)} \]

\[ \sum_i \alpha_i p_i = \frac{1}{\text{vol}(P^\circ)} \cdot \sum_i \text{vol}(F_i^\circ) \frac{p_i}{\| p_i \|} = 0. \]

\[ \sum_i \text{vol(face}_i \cdot \text{normal}_i = 0 \]
let $\mu_P$ be the uniform measure on a polytope $P^o \subset \mathbb{R}^d$.

compute its moments:

$$m_I := \int_{P^o} x^I \, dx = \int_{P^o} x_{i_1}^{i_1} \cdots x_{i_d}^{i_d} \, dx, \quad I = \{i_1 < \cdots i_d\} \in \mathbb{N}^d$$

compute the moment generating function:

$$\sum_{I \in \mathbb{N}^d} \frac{(\sum I + d)!}{I!} m_I t^I.$$

\[ \implies \] this is a rational function whose numerator is the adjoint polynomial of $P$. 

WACHSPRESS FROM ALGEBRAIC STATISTICS

(Kohn, Shapiro, Sturmfels; 2020)
WACHSPRESS FROM SPECTRAL GRAPH THEORY

$$\theta \in \text{Spec}(A) \implies u_1, \ldots, u_d \in \text{Eig}_\theta(A)$$

$$\implies \begin{bmatrix} u_1 & \cdots & u_d \end{bmatrix} = \begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{bmatrix} \in \mathbb{R}^{n \times d}$$

$$\text{Spec}(A) = \{ 3^1, \sqrt{5}^3, 1^5, 0^4, (-2)^4, (-\sqrt{5})^3 \}$$
The many faces of Wachspress coordinates

WACHSPRESS FROM SPECTRAL GRAPH THEORY

**Theorem.** (Izmestiev, 2010)

A polytope skeleton is a spectral embedding of the edge graph w.r.t. suitable edge and vertex weights.

weight matrix \( M \in \mathbb{R}^{n \times n} \) ... Izmestiev matrix of \( P \)

**Applications:**

- rigidity of polyhedral frameworks
- relations between polytopal symmetries and edge graph symmetries
- progress on the Hirsch conjecture
  
  (Narayanan, Shah, Srivastava; 2022)
Wachpress from spectral graph theory

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  ([Narayanan, Shah, Srivastava; 2022](#))

$$\alpha_i := \sum_j M_{ij} \quad (W., 2023)$$
The many faces of Wachspress coordinates

**Izmestiev’s Theorem**

**Theorem.** *(Izmestiev, 2007)*

The Izmestiev matrix satisfies

(i) $M_{ij} > 0$ whenever $ij \in E$,

(ii) $M_{ij} = 0$ whenever $i \neq j$ and $ij \notin E$,

(iii) $\dim \ker(M) = d$,

(iv) $MX_P = 0$, where $X_P^\top = (p_1, \ldots, p_n) \in \mathbb{R}^{d \times n}$,

(v) $M$ has a single positive eigenvalue of multiplicity 1. *(Lorentzian)*

**Consequences:**

- Defines a function $P \ni x \mapsto \theta_1(x) > 0$ Where are the extremal values?

- $M$ has a unique strictly positive eigenvector $z \in \mathbb{R}_+^n$ (to $\theta_2$):
  
  $\Rightarrow$ defines GBC’s $P \ni x \mapsto z(x) =: \text{spectral coordinates}$
**Pointed polytopes**

\[ := \text{polytope } P \subset \mathbb{R}^d + \text{point } x_P \in \text{int}(P) \]

We can speak of
- *the* polar dual of a pointed polytope
- *the* Wachspress coordinates of a pointed polytope
- *the* Izmestiev matrix of a pointed polytope
- ...
The many faces of Wachspress coordinates

**WACHSPRESS FROM VARIATION OF VOLUME**

\[ P^\circ(\mathbf{c}) := \{ x \in \mathbb{R}^d \mid \langle x, p_i \rangle \leq c_i \text{ for all } i \in V(G_P) \}. \]

where \( \mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n. \)
The many faces of Wachspress coordinates

**WACHSPRESS FROM VARIATION OF VOLUME**

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where \( c = (c_1, \ldots, c_n) \in \mathbb{R}^n \). Expand \( \text{vol}(P^\circ(c)) \) at \( c = 1 \):

\[ \text{vol}(P^\circ(c)) = \text{vol}(P^\circ) + \langle \tilde{\alpha}, c - 1 \rangle + \frac{1}{2}(c - 1)^\top \tilde{M}(c - 1) + \cdots \]
The many faces of Wachspress coordinates

WACHSPRESS FROM RIGIDITY THEORY

\[ \omega : E \rightarrow \mathbb{R} \]

\[ \forall i \in V : \sum_{j : ij \in E} \omega_{ij} (p_j - p_i) = 0 \]
The many faces of Wachspress coordinates

**WACHSPRESS FROM RIGIDITY THEORY**

\[
\text{stress } \omega : E \rightarrow \mathbb{R}
\]

\[
\forall i \in V: \sum_{j:i,j \in E} \omega_{ij}(p_j - p_i) = 0
\]

**Lemma.**

If \( P \) is simple, then its framework has a unique non-zero stress and

(i) **stresses on the radial bars** (i.e. \( \omega_{0i}, i \in V \)) are Wachspress coordinates

(ii) **stresses on the edge bars** (i.e. \( \omega_{ij}, i,j \in E \)) are Izmestiev matrix entries.
RIGIDITY AND RECONSTRUCTION
**Application: rigidity and reconstruction**

**Theorem.** (W., 2023)

A pointed polytope is uniquely determined (up to affine transformation) by its edge graph, edge lengths and Wachspress coordinates.

... across all dimensions and all combinatorial types!

**Question:** is there a relation to the log-Minkowski problem?
Application: Rigidity and Reconstruction

Conjecture

A pointed polytope $P$ is uniquely determined (up to isometry) by its edge-graph, edge lengths and radii.

Implications:

- reconstruction of matroids from base exchange graph
- strengthening of Kirszbraun theorem
- symmetries of a polytope are encoded in edge lengths and radii.
- ...

Using the Izmestiev matrix one can verify the conjecture if... (W., 2023)
**Application: Rigidity and Reconstruction**

**Conjecture**

A pointed polytope $P$ is uniquely determined (up to isometry) by its edge-graph, edge lengths and radii.

**Implications:**

- reconstruction of matroids from base exchange graph
- strengthening of Kirszbraun theorem
- symmetries of a polytope are encoded in edge lengths and radii.
- ...

Using the **Izmestiev matrix** one can verify the conjecture if...

- $P, Q$ centrally symmetric
- $P \approx Q$ (Hausdorff metric)
- $P \simeq Q$ (combinatorially equivalent)
The Wachspress map $\phi: P \rightarrow Q$

The Wachspress map $\phi: P \rightarrow Q$ maps

$$x \in P \quad \mapsto \quad \alpha(x) \in \Delta_n \quad \mapsto \quad \phi(x) := \sum_i \alpha_i(x)q_i \in Q$$

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**The Wachspress map** $\phi: P \rightarrow Q$

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$$x \in P \quad \longrightarrow \quad \alpha(x) \in \Delta_n \quad \longrightarrow \quad \phi(x) := \sum_{i} \alpha_i(x) q_i \in Q$$
**The Wachspress Map** \( \phi : P \to Q \)

The Wachspress map \( \phi : P \to Q \) maps

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x \in P \quad \mapsto \quad \alpha(x) \in \Delta_n \quad \mapsto \quad \phi(x) := \sum_i \alpha_i(x) q_i \in Q
\]

**Question:** Is there always a point \( x \in \text{int}(P) \) with \( \|\phi(x) - x_Q\| \leq \|x - x_P\| \)?
The Wachspress map $\phi: P \rightarrow Q$

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$$x \in P \quad \mapsto \quad \alpha(x) \in \Delta_n \quad \mapsto \quad \phi(x) := \sum_i \alpha_i(x)q_i \in Q$$

**Question:** Is there always a point $x \in \text{int}(P)$ with $\|\phi(x) - x_Q\| \leq \|x - x_P\|$?
Understanding the variety is key
Injectivity of the Wachspress map

Wachspress map: \( x \in P \mapsto \alpha^P(x) \in \Delta_n \mapsto \sum_i \alpha_i^P(x) q_i \in Q \)

Conjecture. The Wachspress map is injective.
**Injectivity of the Wachspress map**

**Conjecture.**

*The Wachspress map is injective.*

- true in dimension $d = 2$.
- open in dimension $d \geq 3$.
- other commonly used GBCs are **not** injective!

Understanding injectivity $=$ understanding secant directions of $V(P)$
**Wachspress ideals vs. Stanley-Reisner ideals**

$P$ ... **simplicial** polytope

Wachspress variety

\[ V(P) \cap \partial \Delta_n \simeq \partial P \]
Understanding the Wachspress variety

**Wachspress ideals vs. Stanley-Reisner ideals**

\( P \ldots \textbf{simplicial} \text{ polytope} \)

Wachspress variety

\[ V(P) \cap \partial \Delta_n \simeq \partial P \]
Understanding the Wachspress variety

**Wachspress ideals vs. Stanley-Reisner ideals**

$P$ ... **simplicial** polytope

$$V(P) \cap \partial \Delta_n \cong \partial P$$

**Observation:**

- $I(P) = \langle f_1, f_2, \ldots \rangle$
- the monomials of $f_i$ correspond to the non-faces of $P$. 
**Wachspress ideals vs. Stanley-Reisner ideals**

Let $P$ be a **simplicial** polytope.

The Wachspress variety can be described as:

$$V(P) \cap \partial \Delta_n \simeq \partial P$$

**Observation:**

- $I(P) = \langle f_1, f_2, \ldots \rangle$
- The monomials of $f_i$ correspond to the non-faces of $P$. 

Some relation:

Wachspress ideal $\sim$ Stanley-Reisner ideal
Theorem. (Irving, Schenck, 2013)

For polygons ($d = 2$) holds

- the initial ideal of the Wachspress ideal (using graded lex order) is given by the Stanley-Reisner ideal.
- the Wachspress variety is
  - arithmetically Cohen-Macaulay,
  - of Castelnuovo-Mumford regularity two.

Question: How does this generalize to $d \geq 3$?
Deciding polytopality of simplicial spheres

\[ S \subset \partial \Delta_n \quad \text{... d-dimensional simplicial sphere} \]

**Task:** find a variety \( V \subset \Delta_n \) so that ...

- \( V \cap \partial \Delta_n = S \).
  - \( I(V) \) is generated by polynomials using minimal non-faces.
- the graph of a rational function of degree \( m - d \).
- smooth inside of \( \Delta_n \).
- ...

If **No**, then \( S \) is **not** polytopal!
Thank you.

M. Winter, “Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints” (2023)