Wachspress Objects and the Reconstruction of Convex Polytopes

from Partial Data

Martin Winter

University of Warwick

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The setting: convex polytopes

\[ P = \text{conv}\{p_1, \ldots, p_n\} \subset \mathbb{R}^d \]

- always convex
- general dimension \( d \geq 2 \)
- general geometry & combinatorics (not only simple/simplicial/lattice/...)
- always of full dimension
**Combinatorics of polytopes**

**edge-graph** ... \( G_P := \{ \text{vertices and edges of } P \} \)

**skeleton** ... embedding \( p : G_P \to \mathbb{R}^d \) of the edge-graph

**face lattice** ... \( \mathcal{F}(P) := \{ \text{faces of } P \text{ ordered by inclusion} \} \)

or combinatorial type
Wachspress objects
WACHSPRESS OBJECTS

“A family of objects that appear as bridges between algebra, geometry and combinatorics”

- Wachspress coordinates
- Wachspress variety
- Wachspress ideal
- Wachspress map
- adjoint polynomial
- adjoint hypersurface
- Izmestiev matrix
- ...

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Wachspress objects

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**GENERALIZED BARYCENTRIC COORDINATES**

\[
\{(\alpha_1, ..., \alpha_n) \in \mathbb{R}_{\geq 0}^n \mid \alpha_1 + \cdots + \alpha_n = 1\}\]

Generalized barycentric coordinates (GBCs): \(\alpha : P \rightarrow \Delta_n\) satisfy

\[
\sum_i \alpha_i(x)p_i = x \quad \text{(linear precision)}
\]
GENERALIZED BARYCENTRIC COORDINATES

\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_{\geq 0}^n \mid \alpha_1 + \cdots + \alpha_n = 1 \} \downarrow

Generalized barycentric coordinates (GBCs): \( \alpha : P \rightarrow \Delta_n \) satisfy

\[ \sum_i \alpha_i(x) p_i = x \quad \text{(linear precision)} \]

There are ...

- harmonic coordinates,
- mean value coordinates,
- ...
- Wachspress coordinates (Wachspress 1975; Warren, 1996)
Generalized barycentric coordinates (GBCs): $\alpha : P \rightarrow \Delta_n$ satisfy

$$\sum_i \alpha_i(x)p_i = x \quad \text{(linear precision)}$$

There are ...

- harmonic coordinates,
- mean value coordinates,
- ...
- Wachspress coordinates (Wachspress 1975; Warren, 1996)

... have many non-trivially equivalent definitions
The many faces of Wachspress coordinates

I. Unique rational GBCs of lowest possible degree (Warren, 2003)

\[ \alpha_i(x) = \frac{p_i(x)}{q(x)} \]

where \( q(x) = \sum p_i(x) \) ... adjoint polynomial

- there are not always polynomial GBCs
- degree = \#facets - \( d \)
The many faces of Wachspress coordinates

I. Unique rational GBCs of lowest possible degree (Warren, 2003)

\[ \alpha_i(x) = \frac{p_i(x)}{q(x)} \]

where \( q(x) = \sum_i p_i(x) \) ... adjoint polynomial

- there are not always polynomial GBCs
- degree = \#facets − d
- Wachspress variety
  \[ ... V := \text{im}(\alpha) \subseteq \Delta_n \]
- Wachspress ideal ... \( I(V) \)
  \( \cong \) Stanley-Reisner ideal
II. Relative cone volumes (Ju et al., 2005)

polar dual ... \( P^\circ := \{ x \in \mathbb{R}^d \mid \langle x, p_i \rangle \leq 1 \text{ for all } i \in V(G_P) \} \).

\[ \alpha_i = \frac{\text{vol}(F^\circ_i)}{\|p_i\| \text{ vol}(P^\circ)} \]
The many faces of Wachspress coordinates

III. From spectral embeddings of the edge-graph (W., 2023)

\[ \theta \in \text{Spec}(A) \implies u_1, \ldots, u_d \in \text{Eig}_\theta(A) \]

\[ \implies \begin{bmatrix} u_1 & \cdots & u_d \end{bmatrix} = \begin{bmatrix} p_1 & & \\
\vdots & & \\
p_n & & \\ \end{bmatrix} \in \mathbb{R}^{n \times d} \]
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Colin de Verdière embedding

- A polytope skeleton is a spectral embedding of the edge-graph w.r.t. some weighted adjacency matrix \( M \) (Izmestiev, 2010)

\[ \alpha_i := \sum_j M_{ij} \]
The many faces of Wachspress coordinates

IV. Via a variation of volume

\[ P^\circ(c) := \{ x \in \mathbb{R}^d \mid \langle x, p_i \rangle \leq c_i \text{ for all } i \in V(G_P) \}. \]

where \( c = (c_1, ..., c_n) \in \mathbb{R}^n \).
THE MANY FACES OF WACHSPRESS COORDINATES

IV. Via a variation of volume

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where \( c = (c_1, \ldots, c_n) \in \mathbb{R}^n \). Expand \( \text{vol}(P^\circ(c)) \) at \( c = 1 \):

\[ \text{vol}(P^\circ(c)) = \text{vol}(P^\circ) + \langle \tilde{\alpha}, c - 1 \rangle + \frac{1}{2} (c - 1)^\top \tilde{M} (c - 1) + \cdots \]
Wachsspress objects

WACHSPRESS COORDINATES ACROSS DISCIPLINES

- **adjoint polynomial** $q$ cuts out minimal degree surface that passes through “external non-faces”
- algebraic statistics
  - moment varieties of polytopes
  - Bayesian statistics
- intersection theory (computing Segre classes of monomial schemes)
- $P$ with adjoint polynomial is a **positive geometry** *(cf. the amplituhedron from theoretical physics)*
- has also been defined on polycons and smooth convex bodies
- **Izmestiev matrix** has been used
  - to encode polytopal symmetries in colorings of the edge-graph
  - for progress on the Hirsch conjecture
Reconstruction of polytopes from partial data
"In how far is a polytope determined by partial combinatorial and geometric data, up to isometry, affine transformation or combinatorial equivalence?"
Reconstruction of polytopes

“\textit{In how far is a polytope determined by partial combinatorial and geometric data, up to isometry, affine transformation or combinatorial equivalence?}”

\begin{itemize}
  \item Does the edge-graph determine the combinatorics?
\end{itemize}
"In how far is a polytope determined by partial combinatorial and geometric data, up to isometry, affine transformation or combinatorial equivalence?"

Does the edge-graph determine the combinatorics? **No.**
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▶ Does the edge-graph determine the combinatorics? **No.**
▶ Does combinatorics + edge-lengths determine the geometry?
"In how far is a polytope determined by partial combinatorial and geometric data, up to isometry, affine transformation or combinatorial equivalence?"

- Does the edge-graph determine the combinatorics? **No.**
- Does combinatorics + edge-lengths determine the geometry? **No.**
Flexible polytopes
TWO OPPOSING EFFECTS ...

Simple polytopes:

▶ combinatorics can be reconstructed \((\text{Blind \& Mani; Kalai})\)
▶ geometry cannot be reconstructed

Simplicial polytopes:

▶ geometry can be reconstructed, once combinatorics is known \((\text{Cauchy})\)
▶ combinatorics cannot always be reconstructed (e.g. cyclic polytopes)
Two opposing effects …

Simple polytopes:
- combinatorics can be reconstructed \((\text{Blind & Mani; Kalai})\)
- geometry cannot be reconstructed

Simplicial polytopes:
- geometry can be reconstructed, once combinatorics is known \((\text{Cauchy})\)
- combinatorics cannot always be reconstructed (e.g. cyclic polytopes)

… what additional data is needed to permit a reconstruction?
Reconstruction of Pointed Polytopes

"Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints"
(arXiv:2302.14194, accepted at IMRN)
**Pointed Polytopes**

\[ := \text{polytope } P \subset \mathbb{R}^d + \text{point } x_P \in \mathbb{R}^d \]
**Pointed Polytopes**

\[ := \text{polytope } P \subset \mathbb{R}^d + \text{point } x_P \in \mathbb{R}^d \]

**Conjecture. (W., 2023)**

A pointed polytope \( P \) with \( x_P \in \text{int}(P) \) is uniquely determined (up to isometry) by its edge-graph, edge lengths and radii.

implies e.g. reconstruction of matroids from base exchange graph
Point in the interior is necessary ...

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Conjecture. (W., 2023)

If $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ are pointed polytopes with the same edge-graph and

(i) $x_Q \in \text{int}(Q)$
(ii) edges in $Q$ are at most as long as in $P$,
(iii) radii in $Q$ are at least as large as in $P$,

then $P$ and $Q$ are isometric.

“A polytope cannot become larger if all its edges become shorter.”
Conjecture holds in special cases (W., 2023)

The conjecture holds in the following cases:

I. $Q$ is a small perturbation of $P$
   - one can replace $Q$ by a graph embedding $q : G_P \to \mathbb{R}^d$
   - locally rigid as a framework

II. $P$ and $Q$ are centrally symmetric
   - one can replace $Q$ by a centrally symmetric graph embedding $q : G_P \to \mathbb{R}^e$
   - universally rigid as a centrally symmetric framework

III. $P$ and $Q$ are combinatorially equivalent
   - in particular true for polytope of dimension $d \leq 3$
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The conjecture holds in the following cases:

1. \( Q \) is a small perturbation of \( P \)
   - one can replace \( Q \) by a graph embedding \( q: G_P \to \mathbb{R}^d \)
   - \( \mathcal{R} \) locally rigid as a framework

\( P \subset \mathbb{R}^d \quad Q \subset \mathbb{R}^e \quad q : G_P \to \mathbb{R}^e \)
Conjecture holds in special cases (W., 2023)

The conjecture holds in the following cases:

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III. \( P \) and \( Q \) are combinatorially equivalent
   - in particular true for polytope of dimension \( d \leq 3 \)
**General graph embedding version is false**
**Warmup: simplices**

Let $P, Q \subset \mathbb{R}^d$ be simplices,

(i) $0 \in \text{int}(Q)$,

(ii) edges in $Q$ are at most as long as in $P$.

(iii) radii in $Q$ are at least as large as in $P$.

Therefore $P \simeq Q$. □
**Warmup: simplices**

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*Proof.*

...
Rigidity of pointed polytopes

Warmup: simplices

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(i) $0 \in \text{int}(Q)$,

(ii) edges in $Q$ are at most as long as in $P$.

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Proof. For $\alpha \in \Delta_n$ holds

$$\sum_i \alpha_i \|p_i\|^2 = \left\| \sum_i \alpha_i p_i \right\|^2 + \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|p_i - p_j\|^2$$
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$$\sum_i \alpha_i \|q_i\|^2 = \left\| \sum_i \alpha_i q_i \right\|^2 + \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|q_i - q_j\|^2$$
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\forall (\text{ii})

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\]

\(\land (iii)\) \quad \forall (ii)

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**Warmup: simplices**

$P, Q \subset \mathbb{R}^d$ simplices,

(i) $0 \in \text{int}(Q)$, \quad $0 = \sum_i \alpha_i q_i$ ... convex combination

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\[
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$\land \ (iii) \quad \lor \ (i) \quad \lor \ (ii)$

\[
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WARMUP: SIMPLICES

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$$\sum_i \alpha_i \| p_i \|^2 = \left\| \sum_i \alpha_i p_i \right\|^2 + \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \| p_i - p_j \|^2$$

\[ \equiv (iii) \|

\left\| \sum_i \alpha_i q_i \right\|^2 + \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \| q_i - q_j \|^2$$
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\[
\| (\text{iii}) \quad \| (\text{i}) \quad \| (\text{ii})
\]

\[
\sum_i \alpha_i \|q_i\|^2 = \left\| \sum_i \alpha_i q_i \right\|^2 + \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|q_i - q_j\|^2
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Therefore $P \simeq Q$. □
Warmup: simplices

$P, Q \subset \mathbb{R}^d$ simplices,

(i) $0 \in \text{int}(Q), \quad \Rightarrow \quad 0 = \sum_i \alpha_i q_i$ \ldots convex combination

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Proof. For $\alpha \in \Delta_n$ holds

$$\sum_i \alpha_i \|p_i\|^2 = \left\| \sum_i \alpha_i p_i \right\|^2 + \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|p_i - p_j\|^2$$

\| (iii) \| (i) \| ?? \\forall i ??

$$\sum_i \alpha_i \|q_i\|^2 = \left\| \sum_i \alpha_i q_i \right\|^2 + \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|q_i - q_j\|^2$$

Therefore $P \simeq Q$. \qed
**Expansion of polytopes**

Fix \( \alpha \in \Delta_n := \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{\geq 0} \mid \alpha_1 + \cdots + \alpha_n = 1\} \)

\[ \alpha\text{-expansion: } \|P\|^2_\alpha := \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|p_i - p_j\|^2 \]
Expansion of polytopes

Fix $\alpha \in \Delta_n := \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_{\geq 0}^n \mid \alpha_1 + \cdots + \alpha_n = 1\}$

$\alpha$-expansion: $\|P\|_\alpha^2 := \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|p_i - p_j\|^2$

"If edges shrink, then the expansion decreases."
Expansion of polytopes

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$\alpha$-expansion: $\| P \|_{\alpha}^2 := \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \| p_i - p_j \|^2$

“If edges shrink, then the expansion decreases, if $\alpha$ is chosen suitably.”
**Expansion of Polytopes**

Fix \( \alpha \in \Delta_n := \{(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n_{\geq 0} | \alpha_1 + \cdots + \alpha_n = 1\} \)

\( \alpha \)-expansion: \( \| P \|_\alpha^2 := \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \| p_i - p_j \|^2 \)

"If edges shrink, then the expansion decreases, if \( \alpha \) is chosen suitably."

**Key theorem (W., 2023)**

If \( \alpha \) are the Wachspress coordinates of some interior point of \( P \), and edges in \( q : G_P \rightarrow \mathbb{R}^e \) are not longer than in \( P \), then

\[ \| q \|_\alpha \leq \| P \|_\alpha, \]

with equivalence if and only if \( \alpha \simeq_{\text{affine}} P \).
**Consequences**

**Corollary.**

*A pointed polytope is uniquely determined (up to affine transformations) by its edge-graph, edge lengths and Wachspress coordinates.*

A polytope can be reconstructed in polynomial time (via semidefinite program).
Are we done ... ?

\[ \sum_i \alpha_i \|p_i\|^2 = \left\| \sum_i \alpha_i p_i \right\|^2 + \|P\|_\alpha^2 \]

\[ \sum_i \alpha_i \|q_i\|^2 = \left\| \sum_i \alpha_i q_i \right\|^2 + \|Q\|_\alpha^2 \]
Are we done ... ?

\[
\sum_i \alpha_i \|p_i\|^2 = \left\| \sum_i \alpha_i p_i \right\|^2 + \|P\|_\alpha^2 \\
\sum_i \alpha_i q_i = 0 \\
\sum_i \alpha_i \|q_i\|^2 = \left\| \sum_i \alpha_i q_i \right\|^2 + \|Q\|_\alpha^2
\]

What is \(\alpha\)?

- convex coordinates of the special point in \(Q\)
  
  ... and at the same time ...

- Wachspress coordinates of some point in \(P\)
**Rigidity of pointed polytopes**

**Are we done ... ?**

\[
\sum_i \alpha_i \|p_i\|^2 = \left\| \sum_i \alpha_i p_i \right\|^2 + \|P\|^2
\]

\[
\sum_i \alpha_i q_i = 0
\]

\[
\sum_i \alpha_i \|q_i\|^2 = \left\| \sum_i \alpha_i q_i \right\|^2 + \|Q\|^2
\]

**What is \(\alpha\)?**

- convex coordinates of the special point in \(Q\)
  
  ... and at the same time ...

- Wachspress coordinates of *some* point in \(P\)

**Can we have this?**
The Wachspress map \( \phi : P \to Q \)

\[
\sum_i \alpha_i \|p_i\|^2 = \left\| \sum_i \alpha_i p_i \right\|^2 + \|P\|^2_{\alpha}
\]

\[
\sum_i \alpha_i \|q_i\|^2 = \left\| \sum_i \alpha_i q_i \right\|^2 + \|Q\|^2_{\alpha}
\]

The Wachspress map \( \phi : P \to Q \) maps

\[
x \in P \quad \mapsto \quad \alpha(x) \in \Delta_n \quad \mapsto \quad \phi(x) := \sum_i \alpha_i(x) q_i \in Q
\]
The Wachspress map \( \phi: P \to Q \)

\[
\sum_i \alpha_i \|p_i\|^2 = \|x\|^2 + \|P\|_\alpha^2
\]

\[
\sum_i \alpha_i \|q_i\|^2 = \|\phi(x)\|^2 + \|Q\|_\alpha^2
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The Wachspress map \( \phi: P \to Q \) maps

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The Wachspress map $\phi: P \rightarrow Q$

\[
\sum_i \alpha_i \|p_i\|^2 = \|x\|^2 + \|P\|_\alpha^2
\]
\[
\bigwedge^\aleph \bigvee \bigvee
\]

\[
\sum_i \alpha_i \|q_i\|^2 = \|\phi(x)\|^2 + \|Q\|_\alpha^2
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The Wachspress map $\phi: P \rightarrow Q$ maps

\[
x \in P \quad \longmapsto \quad \alpha(x) \in \Delta_n \quad \longmapsto \quad \phi(x) := \sum_i \alpha_i(x)q_i \in Q
\]

The remaining question: how to find $x \in \text{int}(P)$ with $\|x\| \geq \|\phi(x)\|$?
**Key lemma.**

If $P \subset \mathbb{R}^d$ and $q : G_P \to \mathbb{R}^e$ satisfy

*(i)* there is $x \in \text{int}(P)$ with $\|x\| \geq \|\phi(x)\|$, (e.g. if $\phi(x) = 0$)

*(ii)* edges in $q$ are at most as long as in $P$,

*(iii)* radii in $q$ are at least as large as in $P$,

then $q$ is isometric the skeleton of $P$.
We can have it in special cases ... 

Key lemma.

If \( P \subset \mathbb{R}^d \) and \( q : G_P \rightarrow \mathbb{R}^e \) satisfy

(i) there is \( x \in \text{int}(P) \) with \( \|x\| \geq \|\phi(x)\| \), (e.g. if \( \phi(x) = 0 \))

(ii) edges in \( q \) are at most as long as in \( P \),

(iii) radii in \( q \) are at least as large as in \( P \),

then \( q \) is isometric the skeleton of \( P \).

Resolved special cases:

- \( P \) and \( q \) centrally symmetric \( (\phi(0) = 0) \)
- \( q \) a small perturbation of \( P \)'s skeleton \( (0 \in B_{\epsilon}(0) \subset P \rightarrow 0 \in \phi(B_{\epsilon}(0))) \)
- \( P \) and \( Q \) combinatorially equivalent \( (\phi : P \rightarrow Q \) is surjective)
When the Wachspress map condition fails ...
Using Wachspress coordinates and Izmestiev matrix
**Recalling the statement**

**Key theorem (W., 2023)**

If $\alpha$ are the Wachspress coordinates of some interior point of $P$, and edges in $q : G_P \rightarrow \mathbb{R}^e$ are not longer than in $P$, then

$$\|q\|_\alpha \leq \|P\|_\alpha.$$

"The skeleton of $P$ has the maximal $\alpha$-expansion among all embeddings of $G_P$ whose edges are not longer than in $P."$

$$\max_q \|q\|_\alpha \quad \text{s.t.} \quad \|q_i - q_j\| \leq \|p_i - p_j\|, \quad \text{for all } i, j \in E$$

$$q_1, \ldots, q_n \in \mathbb{R}^n$$
Proof via semidefinite programming

\[
\begin{align*}
\max \quad & \|q\|_\alpha \\
\text{s.t.} \quad & \|q_i - q_j\| \leq \|p_i - p_j\|, \quad \text{for all } ij \in E \\
& q_1, \ldots, q_n \in \mathbb{R}^n
\end{align*}
\]
**Proof via Semidefinite Programming**

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\max & \quad \|q\|_\alpha \\
\text{s.t.} & \quad \|q_i - q_j\| \leq \|p_i - p_j\|, \quad \text{for all } ij \in E \\
& \quad q_1, \ldots, q_n \in \mathbb{R}^n \\
\downarrow & \quad \text{by translation invariance} \\
\max & \quad \sum_i \alpha_i \|q_i\|^2 \\
\text{s.t.} & \quad \sum_i \alpha_i q_i = 0 \\
& \quad \|q_i - q_j\| \leq \|p_i - p_j\|, \quad \text{for all } ij \in E \\
& \quad q_1, \ldots, q_n \in \mathbb{R}^n
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\end{align*}
\]

\[\downarrow \quad \text{dual program}\]

\[
\begin{align*}
\min & \quad \sum_{ij \in E} w_{ij} \|p_i - p_j\|^2 \\
\text{s.t.} & \quad L_w - \text{diag}(\alpha) + \mu \alpha \alpha^\top \succeq 0 \\
& \quad w \geq 0, \mu \text{ free}
\end{align*}
\]
**Proof via Semidefinite Programming**

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\begin{align*}
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\text{s.t.} & \quad \|q_i - q_j\| \leq \|p_i - p_j\|, \quad \text{for all } ij \in E
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\]

\[q_1, \ldots, q_n \in \mathbb{R}^n\]

\[\downarrow \quad \text{by translation invariance}\]

\[
\|P\|_\alpha^2 = \max \quad \sum_i \alpha_i \|p_i\|^2
\]

\[\text{s.t.} \quad \sum_i \alpha_i q_i = 0\]

\[
\|q_i - q_j\| \leq \|p_i - p_j\|, \quad \text{for all } ij \in E
\]

\[q_1, \ldots, q_n \in \mathbb{R}^n\]

\[\downarrow \quad \text{dual program}\]

\[
\|P\|_\alpha^2 = \min \quad \sum_{ij \in E} M_{ij} \|p_i - p_j\|^2
\]

\[\text{s.t.} \quad Lw - \text{diag}(\alpha) + \mu \alpha \alpha^\top \succeq 0\]

\[w \geq 0, \mu \text{ free}\]
IZMESTIEV’S THEOREM

Theorem. (IZMESTIEV, 2007)

The Izmestiev matrix satisfies

(i) \( M_{ij} > 0 \) whenever \( ij \in E \),
(ii) \( M_{ij} = 0 \) whenever \( i \neq j \) and \( ij \notin E \),
(iii) \( \dim \ker(M) = d \),
(iv) \( MX_P = 0 \), where \( X_P^\top = (p_1, \ldots, p_n) \in \mathbb{R}^{d \times n} \),
(v) \( M \) has a single positive eigenvalue of multiplicity 1.

\[
\sum_{ij \in E} M_{ij} \| p_i - p_j \|^2 = \frac{1}{2} \sum_{i,j} M_{ij} \| p_i - p_j \|^2
\]

\[
= \sum_i \left( \sum_j M_{ij} \right) \| p_i \|^2 - \sum_{i,j} M_{ij} \langle p_i, p_j \rangle
\]

\[
= \sum_i \alpha_i \| p_i \|^2 - \text{tr}(MX_P X_P^\top) = \sum_i \alpha_i \| p_i \|^2 = \| P \|^2_{\alpha}.
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**Theorem.** *(IZMESTIEV, 2007)*

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\[
\sum_{ij \in E} M_{ij} \|p_i - p_j\|^2 = \frac{1}{2} \sum_{i,j} M_{ij} \|p_i - p_j\|^2 = \sum_i \left( \sum_j M_{ij} \right) \|p_i\|^2 - \sum_{i,j} M_{ij} \langle p_i, p_j \rangle = \sum_i \alpha_i \|p_i\|^2 \quad \text{tr}(MX_P X_P^\top) = \sum_i \alpha_i \|p_i\|^2 = \|P\|_\alpha^2.
\]
“Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints”
(arXiv:2302.14194, accepted at IMRN)