Wachspress Objects and the Reconstruction of Convex Polytopes from Partial Data

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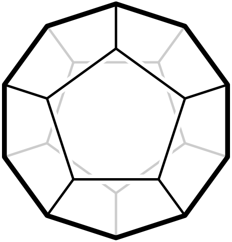
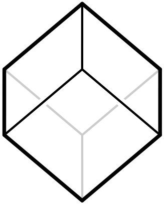
The setting: convex polytopes

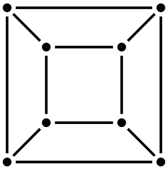
$$P = \operatorname{conv}\{p_1, ..., p_n\} \subset \mathbb{R}^d$$

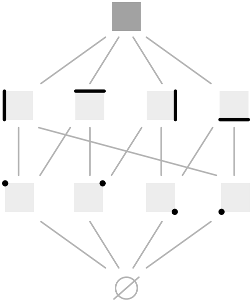


- always convex
- general dimension $d \geq 2$
- general geometry & combinatorics (not only simple/simplicial/lattice/...)
- always of full dimension

Combinatorics of polytopes







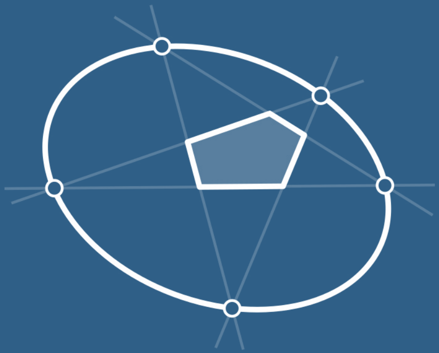
edge-graph ... $G_P := \{\,\text{vertices and edges of } P\,\}$

skeleton ... embedding $p : G_P \to \mathbb{R}^d$ of the edge-graph

face lattice ... $\mathcal{F}(P) := \{\,\text{faces of } P \text{ ordered by inclusion}\,\}$

or combinatorial type

Wachspress objects



Wachspress objects

"A family of objects that appear as bridges between algebra, geometry and combinatorics"

- Wachspress coordinates
- Wachspress variety
- Wachspress ideal
- Wachspress map
- adjoint polynomial
- adjoint hypersurface
- Izmestiev matrix
- ...

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Generalized barycentric coordinates

$$\{(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n_{\geq 0} \mid \alpha_1 + \cdots + \alpha_n = 1)\}$$
$$\downarrow$$

Generalized barycentric coordinates (GBCs): $\alpha : P \to \Delta_n$ satisfy

$$\sum_i \alpha_i(x) p_i = x \quad \text{(linear precision)}$$

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There are ...

- harmonic coordinates,
- mean value coordinates,
- ...
- **Wachspress coordinates** (Wachspress 1975; Warren, 1996)

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There are ...

- harmonic coordinates,
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- ...
- **Wachspress coordinates** (Wachspress 1975; Warren, 1996)

... have many non-trivially equivalent definitions

The many faces of Wachspress coordinates

I. Unique rational GBCs of lowest possible degree (Warren, 2003)

$$\alpha_i(x) = \frac{\mathrm{p}_i(x)}{\mathrm{q}(x)} \qquad \text{where } \mathrm{q}(x) = \sum_i \mathrm{p}_i(x) \text{ ... adjoint polynomial}$$

- there are not always polynomial GBCs
- degree = #facets − d

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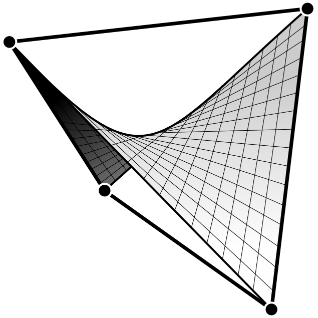
- **Wachspress variety**

  ... $V := \mathrm{im}(\alpha) \subseteq \Delta_n$
- **Wachspress ideal** ... $I(V)$

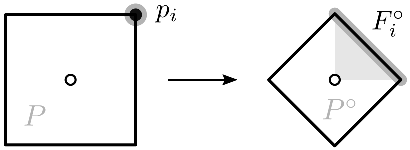
  $\cong$ Stanley-Reisner ideal



The many faces of Wachspress coordinates

II. Relative cone volumes (Ju et al., 2005)

polar dual ... $P^\circ := \{x \in \mathbb{R}^d \mid \langle x, p_i \rangle \le 1$ for all $i \in V(G_P)\}$.



$$\alpha_i = \frac{\mathrm{vol}(F_i^\circ)}{\|p_i\|\,\mathrm{vol}(P^\circ)}$$

The many faces of Wachspress coordinates

III. From spectral embeddings of the edge-graph (W., 2023)

$$\theta \in \mathrm{Spec}(A) \implies u_1, ..., u_d \in \mathrm{Eig}_\theta(A)$$

$$\implies \begin{bmatrix} | & & | \\ u_1 & \cdots & u_d \\ | & & | \end{bmatrix} = \begin{bmatrix} — & p_1 & — \\ & \vdots & \\ — & p_n & — \end{bmatrix} \in \mathbb{R}^{n \times d}$$

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Colin de Verdière embedding

- A polytope skeleton is a ~~spectral embedding~~ of the edge-graph w.r.t. some weighted adjacency matrix M (Izmestiev, 2010)

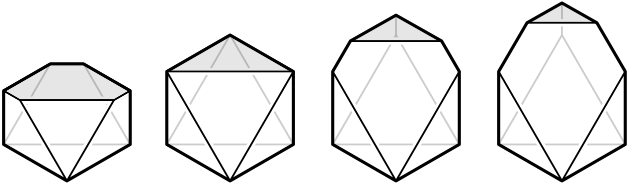
$$\alpha_i := \sum_j M_{ij}$$

The many faces of Wachspress coordinates

IV. Via a variation of volume

$$P^\circ(\mathbf{c}) := \{x \in \mathbb{R}^d \mid \langle x, p_i \rangle \leq c_i \text{ for all } i \in V(G_P)\}.$$

where $\mathbf{c} = (c_1, ..., c_n) \in \mathbb{R}^n$.



THE MANY FACES OF WACHSPRESS COORDINATES

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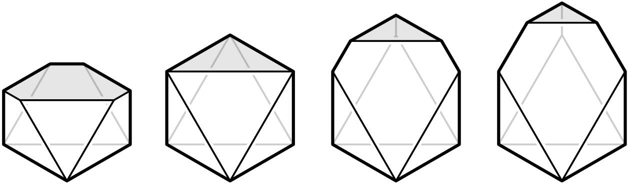
where $\mathbf{c} = (c_1, ..., c_n) \in \mathbb{R}^n$. Expand $\mathrm{vol}(P^\circ(\mathbf{c}))$ at $\mathbf{c} = \mathbf{1}$:

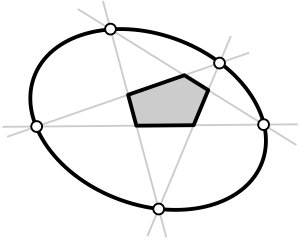
$$\mathrm{vol}(P^\circ(\mathbf{c})) = \mathrm{vol}(P^\circ) + \langle\tilde\alpha, \mathbf{c} - \mathbf{1}\rangle + \tfrac{1}{2}(\mathbf{c} - \mathbf{1})^\top \tilde M(\mathbf{c} - \mathbf{1}) + \cdots$$

↑ Wachspress coordinates

↑ Izmestiev matrix

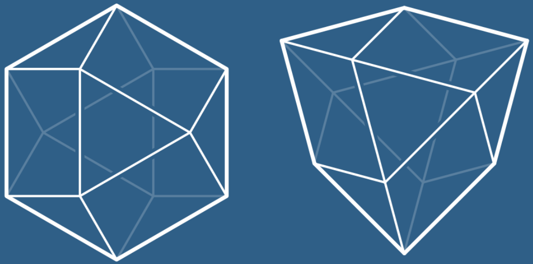


Wachspress coordinates across disciplines



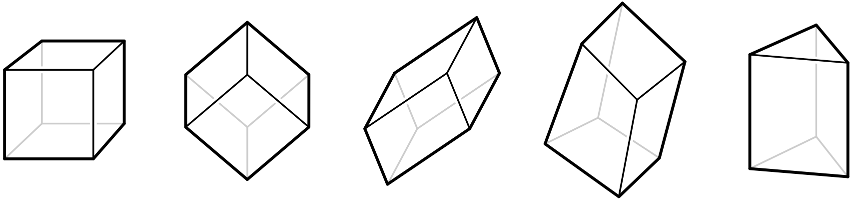
- **adjoint polynomial** q cuts out minimal degree surface that passes through "external non-faces"
- algebraic statistics
  - moment varieties of polytopes
  - Bayesian statistics
- intersection theory (computing Segre classes of monomial schemes)
- P with adjoint polynomial is a **positive geometry** (*cf.* the amplituhedron from theoretical physics)
- has also been defined on polycons and smooth convex bodies
- **Izmestiev matrix** has been used
  - to encode polytopal symmetries in colorings of the edge-gaph
  - for progress on the Hirsch conjecture

Reconstruction of polytopes from partial data



Reconstruction of polytopes

"In how far is a polytope determined by partial combinatorial and geometric data, up to isometry, affine transformation or combinatorial equivalence?"



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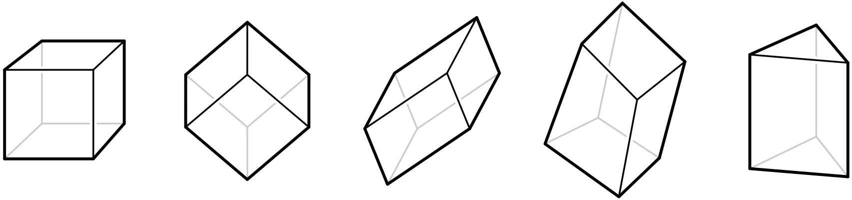

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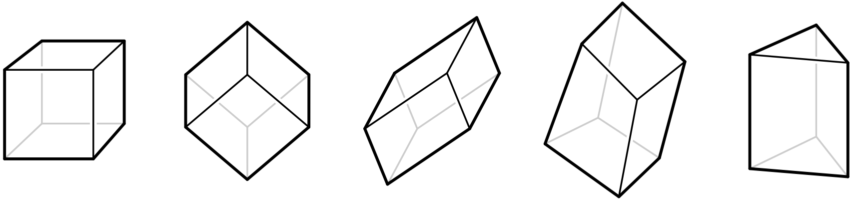

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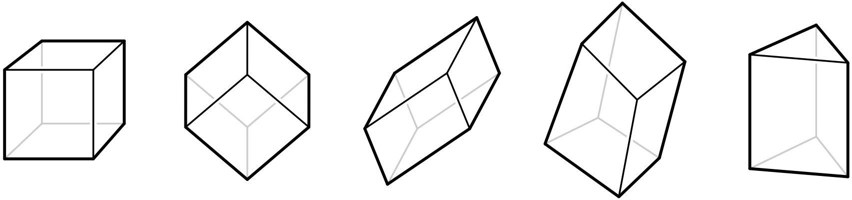

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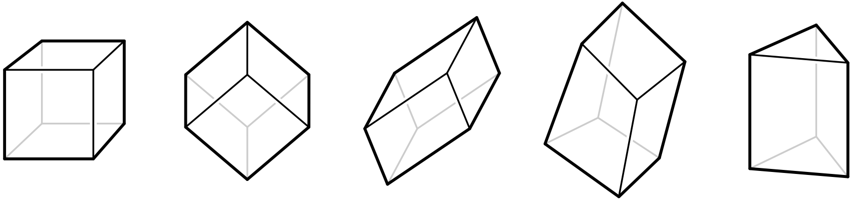

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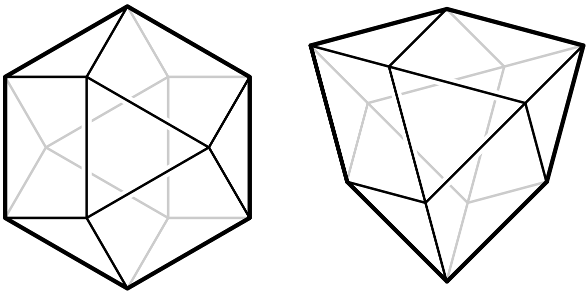
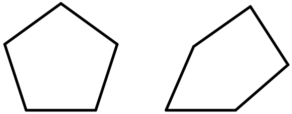
Flexible polytopes



Two opposing effects ...

Simple polytopes:

- combinatorics can be reconstructed (Blind & Mani; Kalai)
- geometry cannot be reconstructed



Simplicial polytopes:

- geometry can be reconstructed, once combinatorics is known (Cauchy)
- combinatorics cannot always be reconstructed (e.g. cyclic polytopes)

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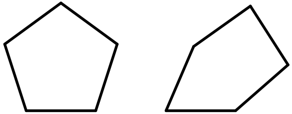

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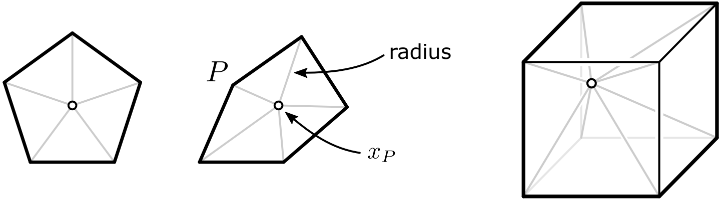
... *what additional data is needed to permit a reconstruction?*

Reconstruction of pointed polytopes

"Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints"
(arXiv:2302.14194, accepted at IMRN)

Pointed polytopes

:= polytope $P \subset \mathbb{R}^d$ + point $x_P \in \mathbb{R}^d$



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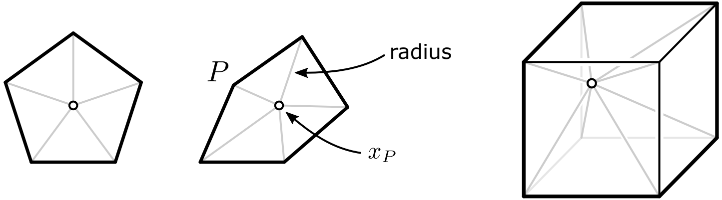

Conjecture. (W., 2023)

A pointed polytope P with $x_P \in \operatorname{int}(P)$ is uniquely determined (up to isometry) by its edge-graph, edge lengths and radii.

implies *e.g.* reconstruction of matroids from base exchange graph

Point in the interior is necessary ...

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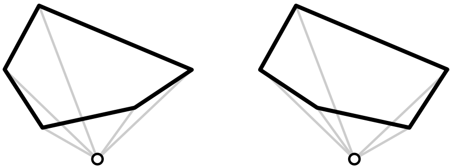


Tensegrity version

Conjecture. (W., 2023)

If $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ are pointed polytopes with the same edge-graph and

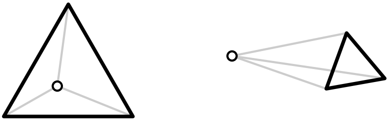
(i) $x_Q \in \text{int}(Q)$

(ii) *edges in Q are <u>at most</u> as long as in P,*

(iii) *radii in Q are <u>at least</u> as large as in P,*

then P and Q are isometric.

"A polytope cannot become larger if all its edges become shorter."



Conjecture holds in special cases (W., 2023)

The conjecture holds in the following cases:

I. Q is a small perturbation of P

- one can replace Q by a graph embedding $q\colon G_P \to \mathbb{R}^d$

$\cong$ locally rigid as a framework

II. P and Q are centrally symmetric

- one can replace Q by a centrally symmetric graph embedding $q\colon G_P \to \mathbb{R}^e$

$\cong$ universally rigid as a centrally symmetric framework

III. P and Q are combinatorially equivalent

- in particular true for polytope of dimension $d \leq 3$

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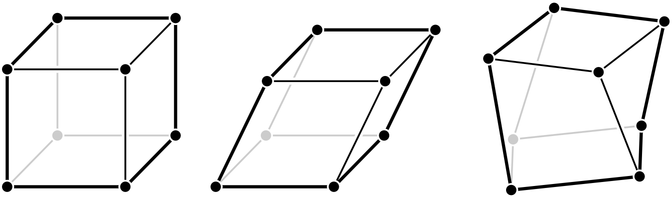

$P \subset \mathbb{R}^d$ $\quad Q \subset \mathbb{R}^e$ $\quad q : G_P \to \mathbb{R}^e$

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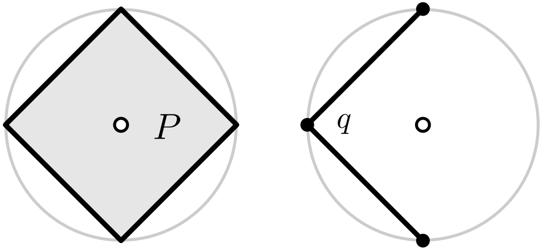
General graph embedding version is false



Warmup: simplices

$P, Q \subset \mathbb{R}^d$ simplices,

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Proof.

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Proof. For $\alpha \in \Delta_n$ holds

$$\sum_i \alpha_i \|p_i\|^2 = \Big\| \sum_i \alpha_i p_i \Big\|^2 + \tfrac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|p_i - p_j\|^2$$

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$$\text{\rotatebox{90}{$\leq$}} \text{ (ii)}$$

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Therefore $P \simeq Q$. □

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$P, Q \subset \mathbb{R}^d$ simplices,

(i) $0 \in \text{int}(Q)$, $\implies 0 = \sum_i \alpha_i q_i$ ... convex combination

(ii) edges in Q are at most as long as in P.

(iii) radii in Q are at least as large as in P.

Proof. For $\alpha \in \Delta_n$ holds

$$\sum_i \alpha_i \|p_i\|^2 = \Big\| \sum_i \alpha_i p_i \Big\|^2 + \tfrac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|p_i - p_j\|^2$$

$$\| \text{ (iii)} \qquad \| \text{ (i)} \qquad \vee\!| \text{ ??}$$

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Therefore $P \simeq Q$. □

Expansion of polytopes

Fix $\alpha \in \Delta_n := \{(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n_{\geq 0} \mid \alpha_1 + \cdots + \alpha_n = 1\}$

α-expansion: $$\|P\|_\alpha^2 := \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|p_i - p_j\|^2$$

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"If edges shrink, then the expansion decreases, if α is chosen suitably."

Key theorem (W., 2023)

If α are the Wachspress coordinates of some interior point of P, and edges in $q : G_P \to \mathbb{R}^e$ are not longer than in P, then

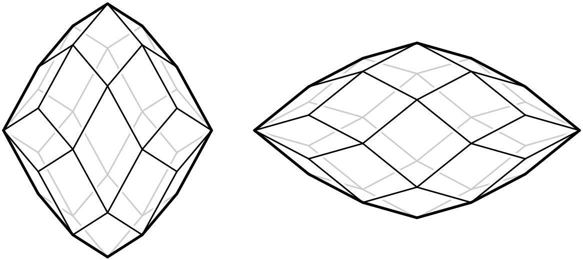
$$\|q\|_\alpha \leq \|P\|_\alpha,$$

with equivalence if and only if $\alpha \simeq_{\text{affine}} P$.

Consequences

Corollary.

A pointed polytope is uniquely determined (up to affine transformations) by its edge-graph, edge lengths and Wachspress coordinates.



A polytope can be reconstructed in polynomial time (via semidefinite program).

Are we done ... ?

$$\sum_i \alpha_i \|p_i\|^2 = \Big\| \sum_i \alpha_i p_i \Big\|^2 + \|P\|_\alpha^2$$

$$\begin{array}{ccc} \wedge\!\mid & \vee\!\mid & \vee\!\mid \end{array}$$

$$\sum_i \alpha_i \|q_i\|^2 = \Big\| \sum_i \alpha_i q_i \Big\|^2 + \|Q\|_\alpha^2$$

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What is α?

- convex coordinates of the special point in Q

  ... and at the same time ...

- Wachspress coordinates of *some* point in P

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Can we have this?

The Wachspress map $\phi\colon P \to Q$

$$\sum_i \alpha_i \|p_i\|^2 = \Big\| \sum_i \alpha_i p_i \Big\|^2 + \|P\|_\alpha^2$$

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The **Wachspress map** $\phi : P \to Q$ maps

$$x \in P \;\longmapsto\; \alpha(x) \in \Delta_n \;\longmapsto\; \phi(x) := \sum_i \alpha_i(x) q_i \in Q$$

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The remaining question: how to find $x \in \mathrm{int}(P)$ with $\|x\| \geq \|\phi(x)\|$?

We can have it in special cases ...

Key lemma.

If $P \subset \mathbb{R}^d$ *and* $q : G_P \to \mathbb{R}^e$ *satisfy*

(i) *there is* $x \in \operatorname{int}(P)$ *with* $\|x\| \geq \|\phi(x)\|$, (*e.g. if* $\phi(x) = 0$)

(ii) *edges in* q *are* *at most* *as long as in* P,

(iii) *radii in* q *are* *at least* *as large as in* P,

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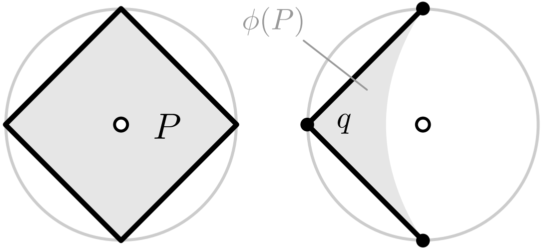
Resolved special cases:

- P and q centrally symmetric ($\phi(0) = 0$)
- q a small perturbation of P's skeleton ($0 \in B_\epsilon(0) \subset P \longrightarrow 0 \in \phi(B_\epsilon(0))$)
- P and Q combinatorially equivalent ($\phi\colon P \to Q$ is surjective)

When the Wachspress map condition fails ...



Using Wachspress coordinates and Izmestiev matrix

Recalling the statement

Key theorem (W., 2023)

If α are the Wachspress coordinates of some interior point of P, and edges in $q : G_P \to \mathbb{R}^e$ are not longer than in P, then

$$\|q\|_\alpha \leq \|P\|_\alpha.$$

"The skeleton of P has the maximal α-expansion among all embeddings of G_P whose edges are not longer than in P."

$$\begin{aligned} \max \quad & \|q\|_\alpha \\ \text{s.t.} \quad & \|q_i - q_j\| \leq \|p_i - p_j\|, \quad \text{for all } ij \in E \\ & q_1, ..., q_n \in \mathbb{R}^n \end{aligned}$$

Proof via semidefinite programming

$$\begin{aligned} \max \quad & \|q\|_\alpha \\ \text{s.t.} \quad & \|q_i - q_j\| \le \|p_i - p_j\|, \quad \text{for all } ij \in E \\ & q_1, ..., q_n \in \mathbb{R}^n \end{aligned}$$

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$\Downarrow$ by translation invariance

$$\begin{aligned} \max \quad & \textstyle\sum_i \alpha_i \|q_i\|^2 \\ \text{s.t.} \quad & \textstyle\sum_i \alpha_i q_i = 0 \\ & \|q_i - q_j\| \le \|p_i - p_j\|, \quad \text{for all } ij \in E \\ & q_1, ..., q_n \in \mathbb{R}^n \end{aligned}$$

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$\Downarrow$ dual program

$$\begin{aligned} \min \quad & \textstyle\sum_{ij \in E} w_{ij} \|p_i - p_j\|^2 \\ \text{s.t.} \quad & L_w - \operatorname{diag}(\alpha) + \mu \alpha \alpha^\top \succeq 0 \\ & w \ge 0, \mu \text{ free} \end{aligned}$$

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$$\begin{aligned} \|P\|_\alpha^2 = \max \quad & \textstyle\sum_i \alpha_i \|p_i\|^2 \\ \text{s.t.} \quad & \textstyle\sum_i \alpha_i q_i = 0 \\ & \|q_i - q_j\| \le \|p_i - p_j\|, \quad \text{for all } ij \in E \\ & q_1, ..., q_n \in \mathbb{R}^n \end{aligned}$$

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Izmestiev's theorem

Theorem. (Izmestiev, 2007)

The Izmestiev matrix satisfies

(i) $M_{ij} > 0$ *whenever* $ij \in E$,

(ii) $M_{ij} = 0$ *whenever* $i \neq j$ *and* $ij \notin E$,

(iii) $\dim\ker(M) = d$,

(iv) $MX_P = 0$, *where* $X_P^\top = (p_1, ..., p_n) \in \mathbb{R}^{d\times n}$,

(v) M *has a single positive eigenvalue of multiplicity 1.*

$$\sum_{ij\in E} M_{ij}\|p_i - p_j\|^2 = \tfrac{1}{2}\sum_{i,j} M_{ij}\|p_i - p_j\|^2$$

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$$= \sum_i \alpha_i\|p_i\|^2 - \mathrm{tr}(\underbrace{MX_P}_{=0}X_P^\top) = \sum_i \alpha_i\|p_i\|^2 = \|P\|_\alpha^2.$$

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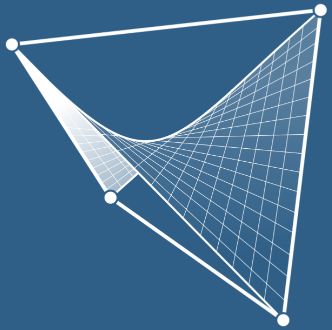
Tack !



"Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints"
(arXiv:2302.14194, accepted at IMRN)