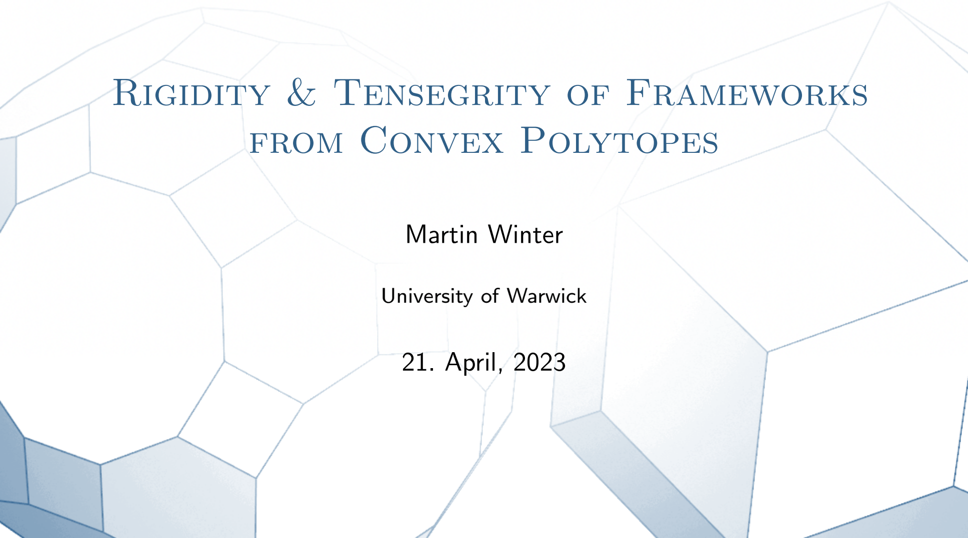


Rigidity & Tensegrity of Frameworks from Convex Polytopes

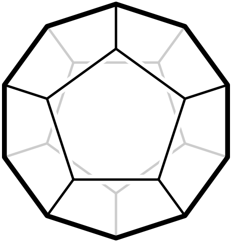
Martin Winter

University of Warwick

21. April, 2023

THE SETTING: CONVEX POLYTOPES

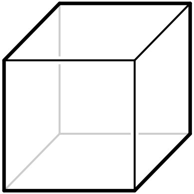
$$P = \mathrm{conv}\{p_1, ..., p_n\} \subset \mathbb{R}^d$$



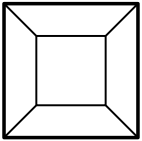
- mostly not simplicial!
- general dimension $d \geq 2$.

General theme: build a framework from the polytope → is it rigid?

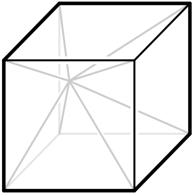
THREE TYPES OF FRAMEWORKS



flexible polytopes



Schlegel diagrams



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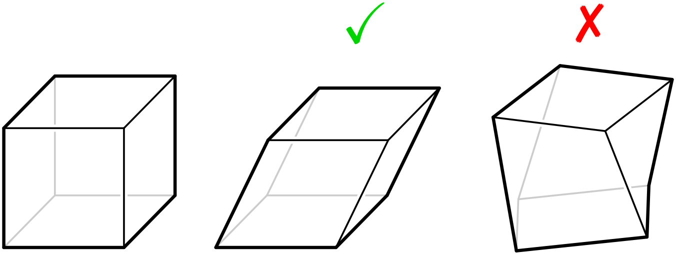
Flexible polytopes

Flexing a polytope



- preserving edge lengths

  but also

- preserve planarity of faces
- preserve convexity
- preserve combinatorial type

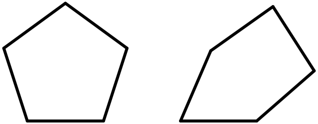
FLEXING A POLYTOPE

Examples of *rigid* polytopes:

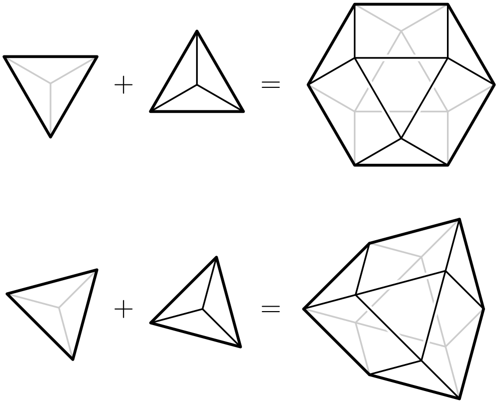
- every simplicial polytope (by Cauchy's rigidity theorem)
- every polytope with triangular 2-faces.

Examples of *flexible* polytopes:

- polygons
- cubes and other prisms

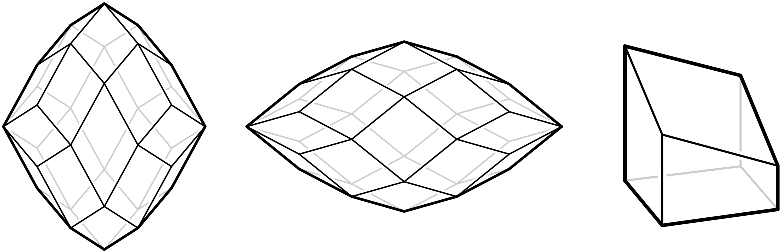


MINKOWSKI SUMS



This includes all **zonotopes**

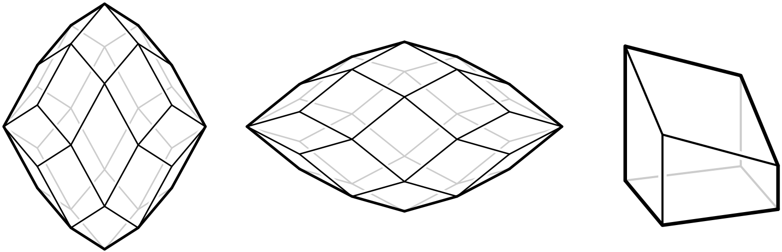
Affine flexes := a flex realized by an affine transformation



Theorem. (Connelly, Gortler, Theran, 2018)

A framework has an affine flex $\iff$ *its edge directions lie on a conic at* ∞.

AFFINE FLEXES := A FLEX REALIZED BY AN AFFINE TRANSFORMATION



Theorem. (CONNELLY, GORTLER, THERAN, 2018)

A framework has an affine flex $\iff$ *its edge directions lie on a conic at* ∞.

For example, if there are at most 5 edge directions (in 3D).

SUMMARIZING ...

We know the following classes of flexible polytopes

- polygons
- Minkowski sums
- all edges on a conic at ∞ (e.g. at most five edge directions)

Question: Are there any others?

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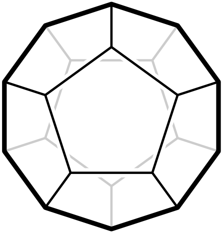
Some other facts:

- $d = 3$: DOF minus constraints $= 0$
- can be transformed into a pure framework (no coplanarity constraints)
- for all known cases, flexibility is preserved under affine transformations

TOY EXAMPLE: THE REGULAR DODECAHEDRON



Question: Is the regular dodecahedron rigid?

- probably, but *we don't know*
- 5-dimensional space of non-trivial infinitesimal flexes
- infinitesimal flexes vanish for any linear transformation (that I tried)

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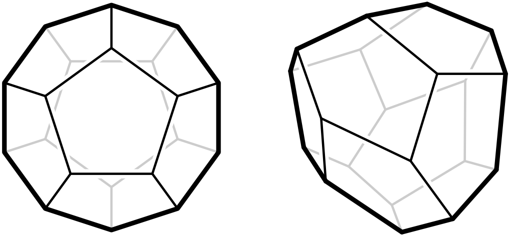

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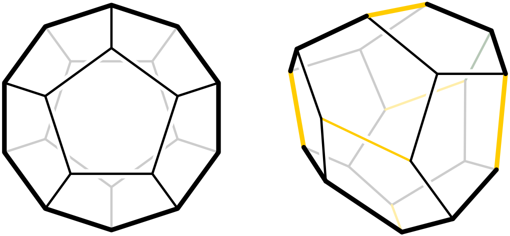

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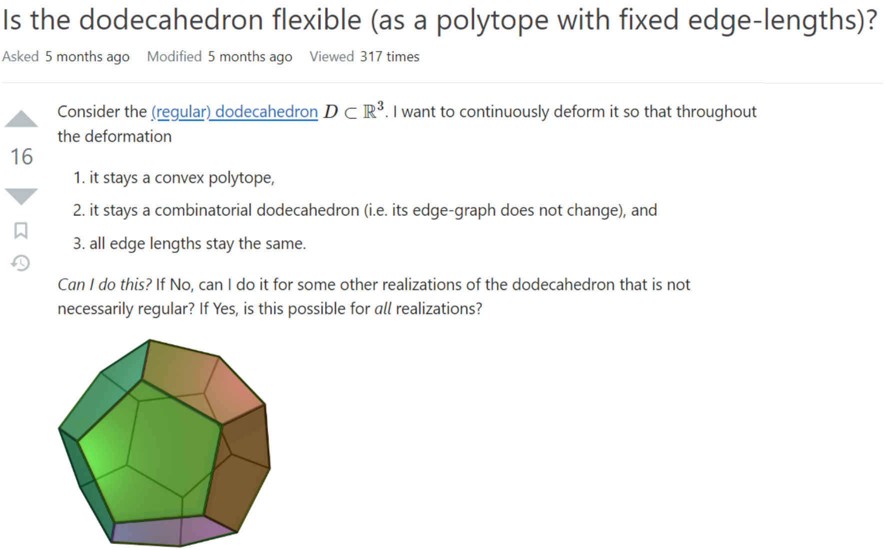
Flexible polytopes



Is the dodecahedron flexible (as a polytope with fixed edge-lengths)?

Asked 5 months ago Modified 5 months ago Viewed 317 times

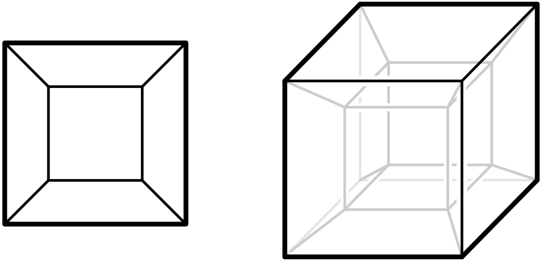
Consider the (regular) dodecahedron $D \subset \mathbb{R}^3$. I want to continuously deform it so that throughout the deformation

1. it stays a convex polytope,
2. it stays a combinatorial dodecahedron (i.e. its edge-graph does not change), and
3. all edge lengths stay the same.

Can I do this? If No, can I do it for some other realizations of the dodecahedron that is not necessarily regular? If Yes, is this possible for *all* realizations?

SCHLEGEL DIAGRAMS

SCHLEGEL DIAGRAM := SPECIAL PROJECTION OF A POLYTOPE



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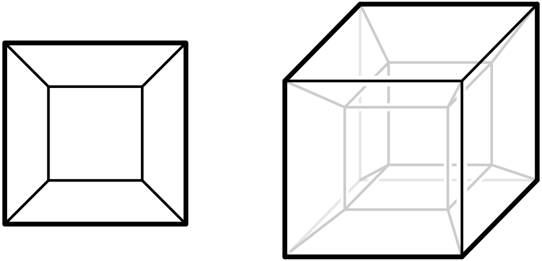
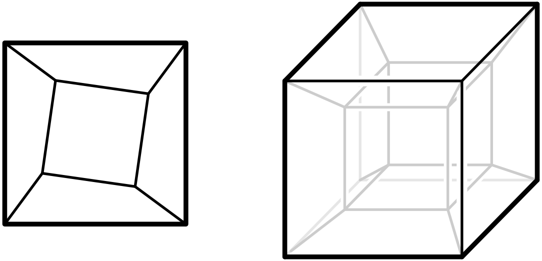


Question: is it always (locally) rigid?

Schlegel diagram := special projection of a polytope



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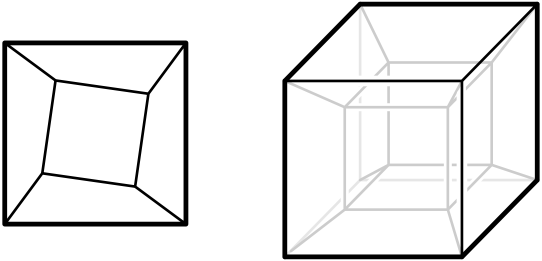
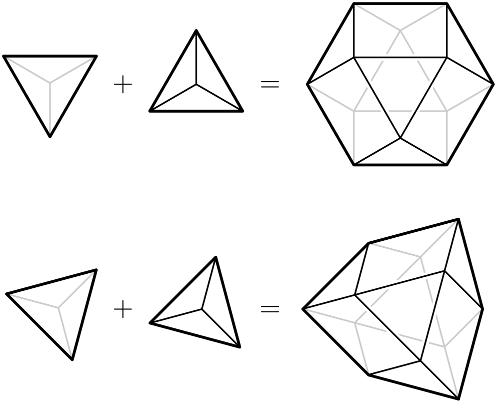


Question: is it always (locally) rigid? **No!**

Flexible Schlegel diagrams



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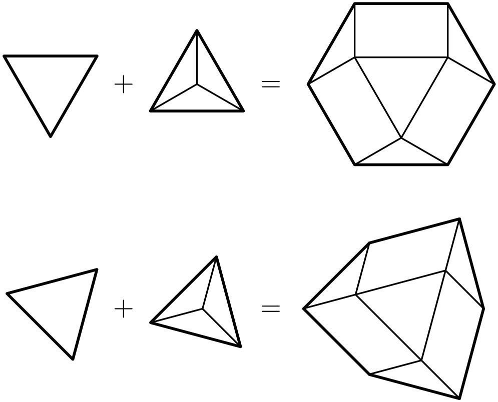

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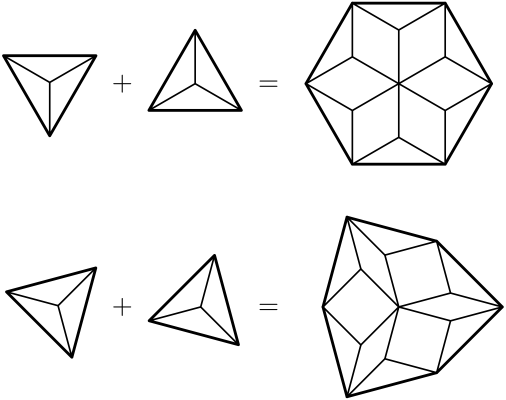

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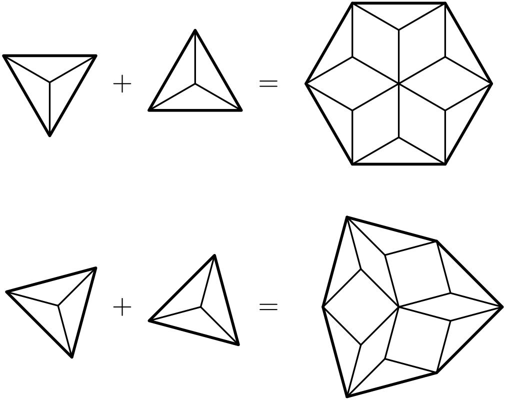
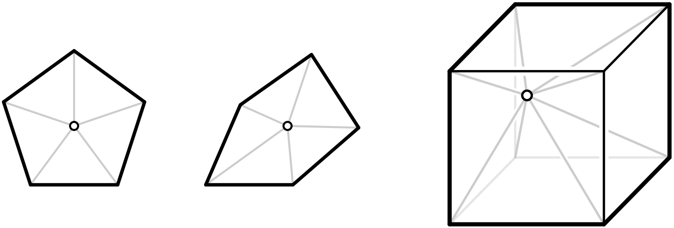


Question: Is flexibility independent of the projection direction?

CONED SKELETA

"Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints"
(`arXiv:2302.14194`)

POINTED POLYTOPES AND CONED SKELETA



- edges as bars + central bars (special case of Schlegel diagram)

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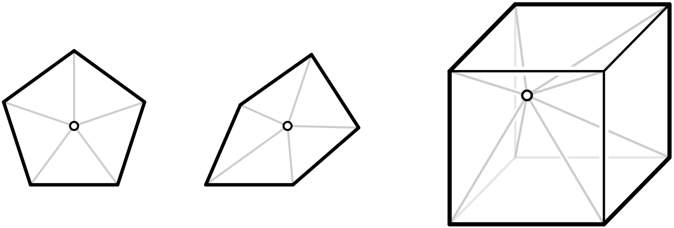

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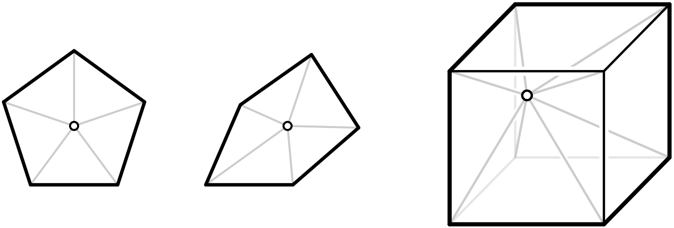

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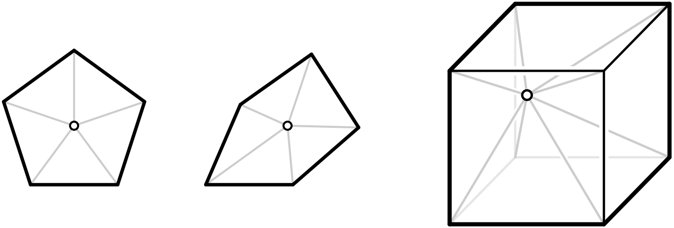

- edges as bars + central bars (special case of Schlegel diagram)
- **tensegrity:** edges as cables + central struts

Question: is it always (locally) rigid? (not always infinitesimally rigid)

RIGIDITY OF CONED SKELETA

Theorem. (W., 2023)

If the point is chosen from the interior of the polytope, then

(i) *as a framework, it is locally rigid* *(actually, prestress stable)*

(ii) *as a polytope, it is globally rigid*

(iii) *as a centrally symmetric framework, it is universally rigid*

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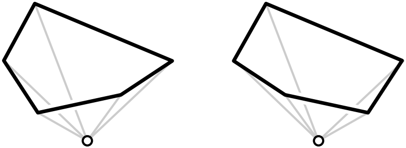

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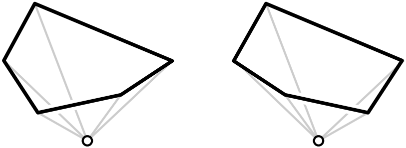

THE MAIN CONJECTURE

Conjecture. (W., 2023)

If $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ are polytopes with the same edge-graph, $x \in Q$ is an _interior_ *point, and if*

(i) *edges in Q are at most as long as in P,*

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then P and Q are isometric.

We proved this in three special cases:

- P and Q are "sufficiently close"
- P and Q are combinatorially equivalent
- P and Q are centrally symmetric

INGREDIENTS TO THE PROOF

convex geometry + spectral graph theory
= “spectral polytope theory”

I. Izmestiev (2007), *“The Colin de Verdière number and graphs of polytopes”*

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$$P^\circ(\mathbf{c}) := \{x \in \mathbb{R}^d \mid \langle x, p_i\rangle \le c_i \text{ for all } i \in V(G_P)\}.$$

Expand $\mathrm{vol}(P^\circ(\mathbf{c}))$ at $\mathbf{c} = \mathbf{1}$:

$$\mathrm{vol}(P^\circ(\mathbf{c})) = \mathrm{vol}(P^\circ) + \langle \underset{\substack{\uparrow\\ \text{Wachspress}\\ \text{coordinates}}}{\tilde\alpha}, \mathbf{c}-\mathbf{1}\rangle + \tfrac{1}{2}(\mathbf{c}-\mathbf{1})^\top \underset{\substack{\uparrow\\ \text{Izmestiev}\\ \text{matrix}}}{\tilde M}(\mathbf{c}-\mathbf{1}) + \cdots$$

These objects define a stress matrix that certifies prestress stability.

FURTHER CONJECTURES

Conjecture.

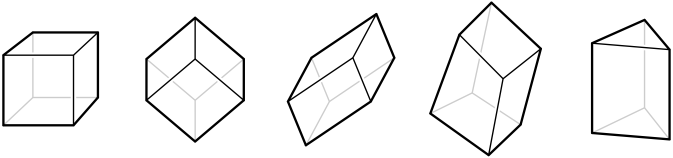
A polytope is determined (up to isometry) by its edge-graph, edge lengths and the distance of each vertex from some common interior point.

Conjecture.

The edge-graph and edge lengths determine the combinatorial type.

Conjecture. (strengthening Cauchy's rigidity theorem)

A polytope is uniquely determined by its 2-skeleton and the shape of its 2-faces.



Thank you.