Rigidity of Polyhedral Spheres beyond Triangulations

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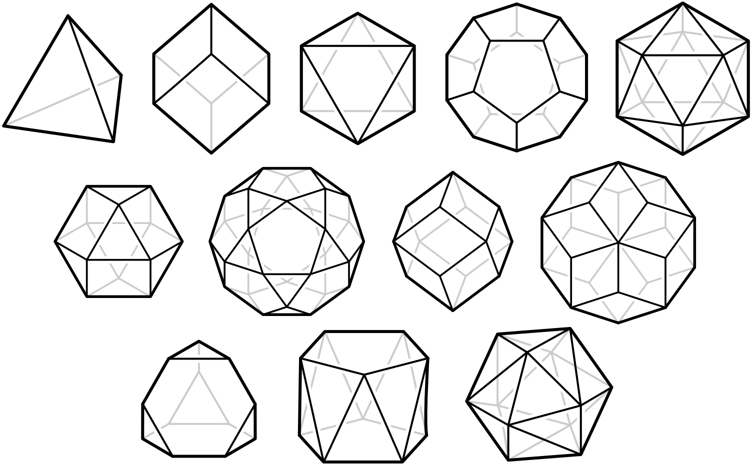
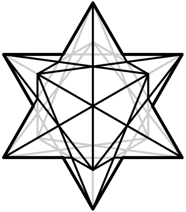
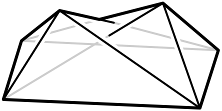
Joint work with Bernd Schulze, Matthias Himmelmann, Albert Zhang

Triangular and polyhedral spheres

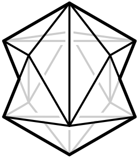


Non-convex and self-intersecting





Bricard octahedron



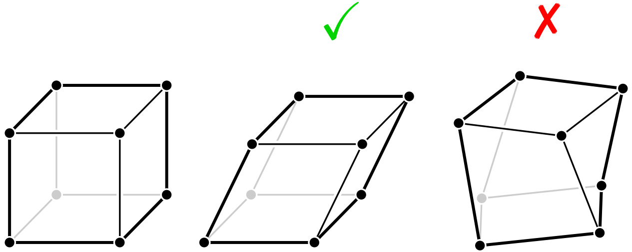
Jessen's icosahedron

Polyhedral spheres

"A polyhedral sphere is a bunch of polygons glued edge to edge so that they form a topological sphere."

- a **polyhedral sphere** $\mathcal{P} = (V, E)$ is a $\overbrace{\text{3-connected planar graph}}^{\text{polyhedral graph}}$.
- its faces we denote by $F_1, ..., F_m \subset V$.
- a **realization** of $\mathcal{P}$ is a map $\boldsymbol{p} : V \to \mathbb{R}^3$ so that the points $p_i, i \in F_k$ lie on a common plane.
- in a **triangulated sphere** all faces are triangles.

Flexing polyhedral spheres

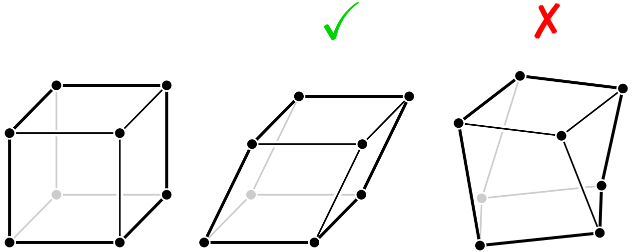


- preserving edge lengths

    but also

- preserve planarity of faces

Flexing polyhedral spheres



- preserving edge lengths

  but also

- preserve planarity of faces

$$\color{blue}{\#\text{DOFs}} - \color{red}{\#\text{constraints}} = \color{blue}{3|V|} - \color{red}{\Big(|E| + \sum_k (|F_k| - 3)\Big)} = 6.$$

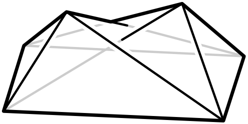
Triangulated Spheres

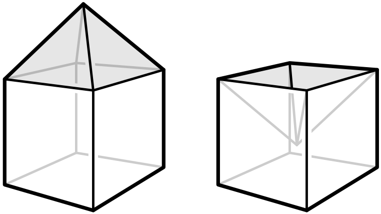
... good old frameworks

Rigidity of triangulated spheres

Core results

- *Convex triangulated spheres are globally rigid.* (Cauchy)
- *Convex triangulated spheres are first-order rigid.* (Dehn)
- *Triangulated spheres are generically first-order rigid.* (Gluck)
- *Flexible triangulated spheres exist.* (Bricard, Connelly, Steffen)



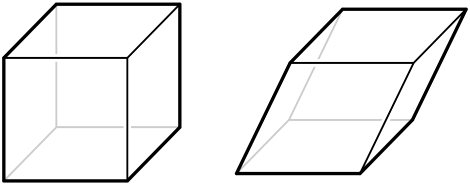


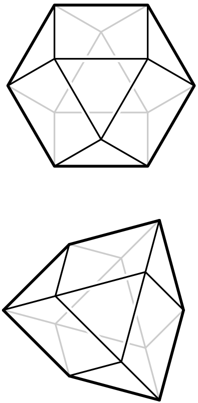
Moving beyond triangulations

Rigidity of general polyhedral spheres

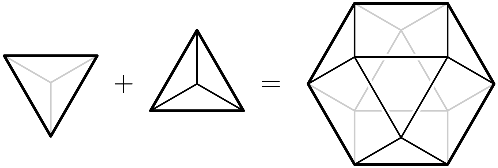
Core results

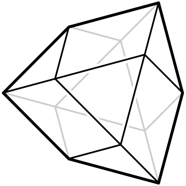
- *Convex polyhedra with fixed face shapes are globally rigid.* (Cauchy)
  (also in higher dimensions) (Alexandrov)
- *Triangulating a convex polyhedron makes it first-order rigid.* (Alexandrov)





Minkowski sums $A + B := \{a + b \mid a \in A, b \in B\}$





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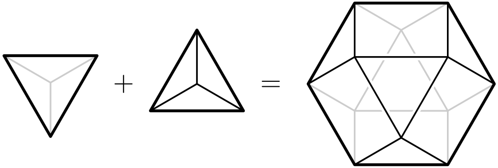
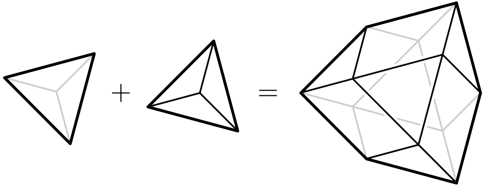


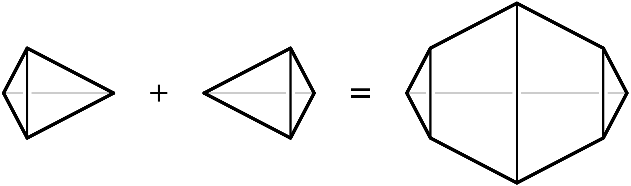


Only Minkowski sums?

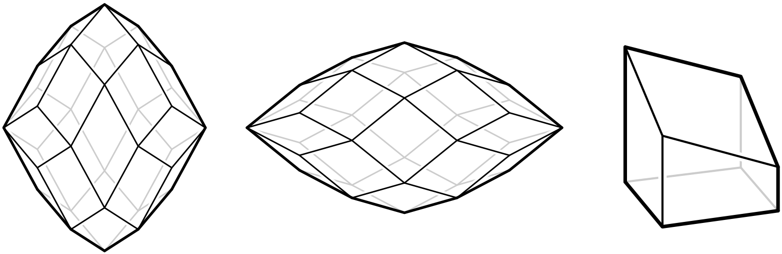
Question: Are all flexible convex polyhedra Minkowski sums?

Notes:

- This includes rotating/flexing a proper Minkowski summand.
- Not all Minkowski sums are flexible.

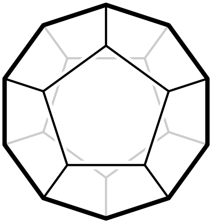


Affine flexes := a flex realized by an affine transformation

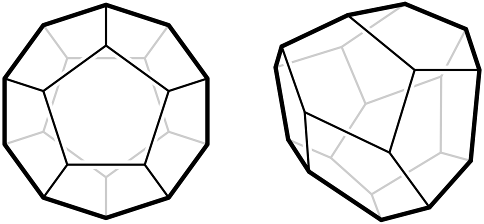


Question: Are all affinely flexible polyhedra Minkowski sums?

Is the regular dodecahedron rigid?



Is the regular dodecahedron rigid?



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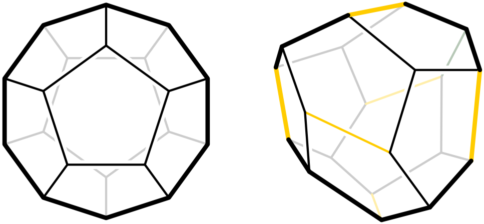

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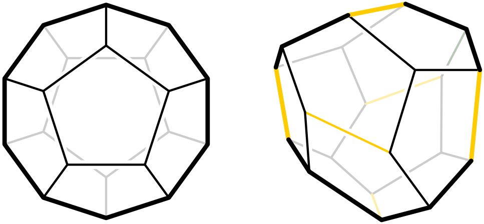
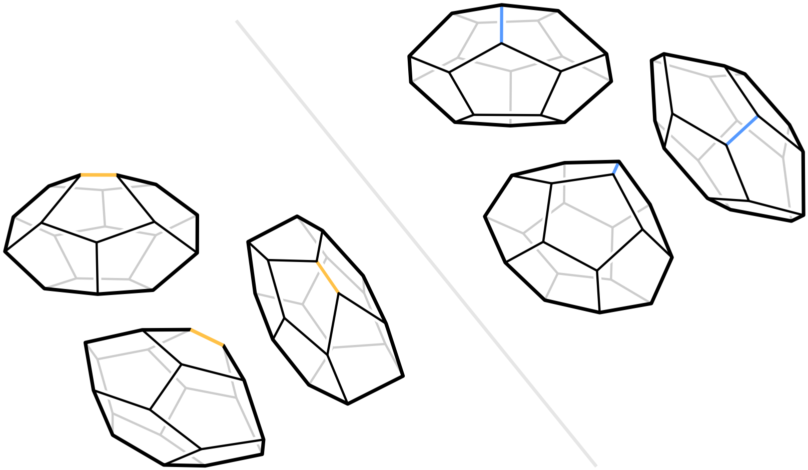


Theorem. (Himmelmann, Schulze, W., Zhang, 2024+)

The regular dodecahedron is ...

- ✗ *not first-order rigid.* (5-dimensional space of first-order flexes)
- ✗ *not prestress stable.*
- ✓ *second-order rigid.*

No generic global rigidity



Many open questions

Question: (about convex spheres)

- Is second order rigidity always sufficient?
- Does flexibility need parallel edges?
- Is polytope rigidity preserved under affine transformations?

  (first-order flexibility is not)

Generic first-order rigidity

Main result

Theorem. (Himmelmann, Schulze, W., Zhang, 2024)

Convex polyhedral spheres are generically first-order rigid.

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$$\mathrm{Real}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{array} \;\middle|\; \langle p_i, n_k \rangle = 1 \;\text{ if } i \in F_k \right\}$$

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$$\textsc{Real-cvx}(\mathcal{P}) := \left\{ \begin{matrix} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{matrix} \;\middle|\; \begin{matrix} \langle p_i, n_k \rangle = 1 & \text{if } i \in F_k \\ \langle p_i, n_k \rangle < 1 & \text{if } i \notin F_k \end{matrix} \right\}$$

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A **finite flex** preserves

$$\begin{aligned} \|p_i - p_j\| &\overset{!}{=} \ell_{ij} = \text{const} && \text{for } ij \in E \\ \langle p_i, n_k \rangle &\overset{!}{=} 1 && \text{for } i \in F_k \end{aligned}$$

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A **first-order flex** $(\dot{\boldsymbol{p}}, \dot{\boldsymbol{n}})$ satisfies

$$\begin{aligned} \langle p_i - p_j, \dot{p}_i - \dot{p}_j \rangle &= 0 && \text{for } ij \in E \\ \langle p_i, \dot{n}_k \rangle + \langle \dot{p}_i, n_k \rangle &= 0 && \text{for } i \in F_k \end{aligned}$$

The Proof

– the triangular case –

"Triangular spheres are generically first-order rigid."

Rigidity matrix $\mathcal{R}(G, \boldsymbol{p})$

$$
\begin{array}{cc}
 & \overbrace{\begin{array}{ccccc} \quad & i \in V & \quad & j \in V & \quad \end{array}}^{3 \times \#\text{vertices}} \\
\#\text{edges} \left\{ \; ij \in E \right. & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & p_i - p_j & \cdots & p_j - p_i & \cdots \\ & \vdots & & \vdots & \end{pmatrix}
\end{array} = \mathcal{R}(G, \boldsymbol{p})
$$

$\dot{\boldsymbol{p}}$ is a first-order flex $\iff \mathcal{R}(G, \boldsymbol{p})\,\dot{\boldsymbol{p}} = 0$

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$\dot{\boldsymbol{p}}$ is a first-order flex $\iff \mathcal{R}(G, \boldsymbol{p})\,\dot{\boldsymbol{p}} = 0$

$(G, \boldsymbol{p})$ is first-order rigid $\iff \operatorname{corank} \mathcal{R}(G, \boldsymbol{p}) = 6$

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\end{array}
$$

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#columns − #rows $= 3|V| - |E| = 6 =$ #trivial first-order flexes.

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$\dot{\boldsymbol{p}}$ is a first-order flex $\iff \mathcal{R}(G, \boldsymbol{p})\,\dot{\boldsymbol{p}} = 0$

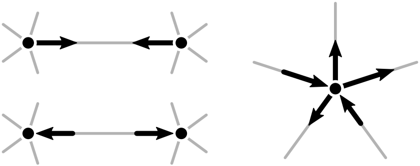
$(G, \boldsymbol{p})$ is first-order rigid $\iff \operatorname{corank} \mathcal{R}(G, \boldsymbol{p})^{\top} = 0$

#columns − #rows $= 3|V| - |E| = 6 =$ #trivial first-order flexes.

Stresses

$$\mathcal{R}(G,\boldsymbol{p})^\top\boldsymbol{\omega} = 0.$$

$$\forall i \in V\colon \sum_{j:ij\in E} \omega_{ij}(p_j - p_i) = 0.$$



first-order flexible $\iff$ $\ker\mathcal{R}(G,\boldsymbol{p})^\top = \{0\}$ $\iff$ $\exists$ non-zero stress

Generic first-order rigidity

$$\textsc{Real}(G) := \{\text{3-dimensional frameworks on } G\} = \mathbb{R}^{3V}$$

$$\begin{aligned}\textsc{Flex}(G) &:= \{\text{first-order flexible frameworks on } G\} \\ &= \{\boldsymbol{p} \in \mathbb{R}^{3V} \mid \operatorname{rank} \mathcal{R}(G,\boldsymbol{p}) < |E|\} \\ &= \{\boldsymbol{p} \in \mathbb{R}^{3V} \mid \det(A) = 0 \text{ for all } |E| \times |E| \text{ submatrices } A \text{ of } \mathcal{R}(G,\boldsymbol{p})\}.\end{aligned}$$

Generic first-order rigidity

$\mathrm{R{\scriptsize EAL}}(G) := \{\,$3-dimensional frameworks on $G\,\} = \mathbb{R}^{3V}$

$\mathrm{F{\scriptsize LEX}}(G) := \{\,$first-order flexible frameworks on $G\,\}$

$$= \{\, \boldsymbol{p} \in \mathbb{R}^{3V} \mid \operatorname{rank} \mathcal{R}(G,\boldsymbol{p}) < |E| \,\}$$

$= \{\, \boldsymbol{p} \in \mathbb{R}^{3V} \mid \det(A) = 0$ for all $|E| \times |E|$ submatrices A of $\mathcal{R}(G,\boldsymbol{p})\,\}$.

$\Longrightarrow$ $\mathrm{F{\scriptsize LEX}}(G) \subseteq \mathbb{R}^{3V}$ is the zero set of polynomials.

$\Longrightarrow$ either $\mathrm{F{\scriptsize LEX}}(G) = \mathbb{R}^{3V}$ or $\mathrm{F{\scriptsize LEX}}(G)$ has measure zero.

Generic first-order rigidity

$\textsc{Real}(G) := \{\,$3-dimensional frameworks on $G\,\} = \mathbb{R}^{3V}$

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$\implies$ $\textsc{Flex}(G) \subseteq \mathbb{R}^{3V}$ is the zero set of polynomials.

$\implies$ either $\textsc{Flex}(G) = \mathbb{R}^{3V}$ or $\textsc{Flex}(G)$ has measure zero.

Recall: a convex realization is first-order rigid. $(\textsc{Dehn})$

$\implies$ $\textsc{Flex}(G) \neq \mathbb{R}^{3V}$.

$\implies$ $\textsc{Flex}(G)$ has measure zero.

The Proof

– the polyhedral case –

"Convex polyhedral spheres are generically first-order rigid."

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$$
\begin{array}{rl}
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\begin{array}{r} \#\text{edges} \ \left\{ \ ij \in E \right. \\ \\ \begin{array}{r} \#\text{vertex-face} \\ \text{incidences} \end{array} \left\{ \ i \in F_k \right. \end{array} &
\left(\begin{array}{cc|c} p_i - p_j & p_j - p_i & \\ & & \\ \hline & & \\ n_k & & p_i \end{array} \right)
\end{array}
$$

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\end{array}
$$

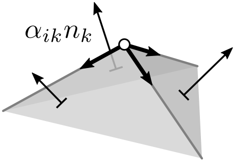
$$\color{blue}{\#\text{columns}} - \color{red}{\#\text{rows}} = \color{blue}{\big(3|V| + 3|F|\big)} - \color{red}{\big(|E| + |VF|\big)} = 6$$

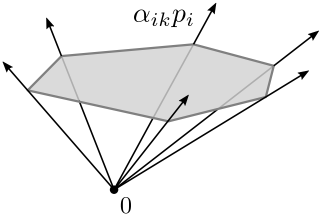
Stresses

$$\mathcal{R}(P)^\top(\boldsymbol{\omega}, \boldsymbol{\alpha}) = 0$$

$$\forall i \in V: \quad 0 = \sum_{j:ij\in E} \omega_{ij}(p_j - p_i) + \sum_{k:i\in F_k} \alpha_{ik} n_k$$

$$\forall k \in F: \quad 0 = \sum_{i\in F_k} \alpha_{ik} p_i$$





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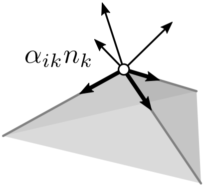

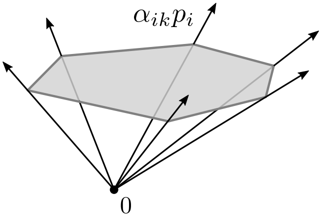

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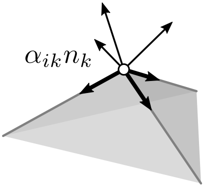

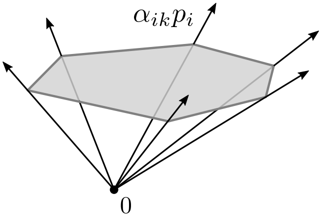

Observation: If F_k is *triangular* face, then $\alpha_{ik} = 0$.

Generic rigidity of polyhedral spheres

Theorem. (Himmelmann, Schulze, W., Zhang)

Convex polyhedral spheres are generically first-order rigid.

$$\forall\,\mathcal{P}\quad \textsc{Flex}(\mathcal{P}) \textit{ has measure zero in } \textsc{Real-cvx}(\mathcal{P}).$$

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Question: What about potentially non-convex polyhedral spheres?

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Question: What about potentially non-convex polyhedral spheres?

Strategy:

1. Polynomial method: $\textsc{Flex}(\mathcal{P}) \subseteq \textsc{Real}(\mathcal{P})$ is a sub-variety.
   $\longrightarrow$ $\textsc{Flex}(\mathcal{P}) = \textsc{Real}(\mathcal{P})$ *or*
   $\textsc{Flex}(\mathcal{P})$ has measure zero in $\textsc{Real}(\mathcal{P})$.
2. Show: there exists at least one realization that is first-order rigid.

Generic rigidity of polyhedral spheres

Theorem. (Himmelmann, Schulze, W., Zhang)

Convex polyhedral spheres are generically first-order rigid.

$$\forall \mathcal{P} \quad \text{Flex}(\mathcal{P}) \textit{ has measure zero in } \text{Real-cvx}(\mathcal{P}).$$

Question: What about potentially non-convex polyhedral spheres?

Strategy:

1. Polynomial method: $\text{Flex}(\mathcal{P}) \subseteq \text{Real-cvx}(\mathcal{P})$ is a sub-variety.
   $\longrightarrow$ $\text{Flex}(\mathcal{P}) = \text{Real-cvx}(\mathcal{P})$ *or*
   $\text{Flex}(\mathcal{P})$ has measure zero in $\text{Real-cvx}(\mathcal{P})$.
2. Show: there exists at least one convex realization that is first-order rigid.

Theorem. (Steinitz)

$\text{Real-cvx}(\mathcal{P}) \subset \mathbb{R}^{3V} \times \mathbb{R}^{3F}$ *is (an open subset of) a smooth, irreducible, contractible variety of dimension* $|E| + 6$.

Reduction to existence

$$\textsc{Real}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{array} \;\middle|\; \langle p_i, n_k \rangle = 1 \;\; \text{if } i \in F_k \right\}$$

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$$\begin{aligned} \textsc{Flex}(\mathcal{P}) &:= \{ (\boldsymbol{p}, \boldsymbol{n}) \in \textsc{Real-cvx}(P) \mid (\mathcal{P}, \boldsymbol{p}, \boldsymbol{n}) \text{ is first-order flexible} \} \\ &= \{ (\boldsymbol{p}, \boldsymbol{n}) \in \textsc{Real-cvx}(P) \mid \operatorname{rank} \mathcal{R}(\mathcal{P}, \boldsymbol{p}, \boldsymbol{n}) < |E| + |VF| \} \end{aligned}$$

$\Longrightarrow$ $\textsc{Flex}(G) \subseteq \textsc{Real-cvx}(P)$ is the zero set of polynomials.

$\Longrightarrow$ either $\textsc{Flex}(\mathcal{P}) = \textsc{Real-cvx}(P)$
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$\implies$ either $\mathrm{Flex}(\mathcal{P}) = \mathrm{Real\text{-}cvx}(P)$
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It remains to show: there exists a first-order rigid convex realization.

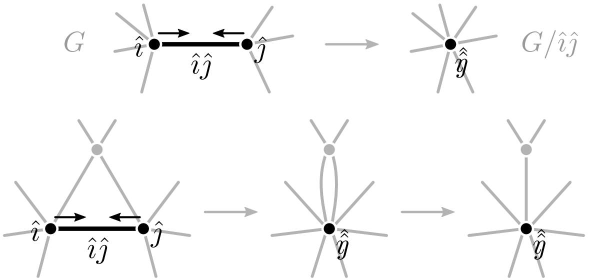
The Proof

– proving existence –

"There exists at least one first-order rigid realization."

Strategy: INDUCTION ON #EDGES

Decreasing the edge number by **contraction**:



Theorem. (TUTTE)

If $G \neq K_4$ is 3-connected, there is an edge $e \in E$ for which G/e is 3-connected.

Strategy: INDUCTION ON #EDGES

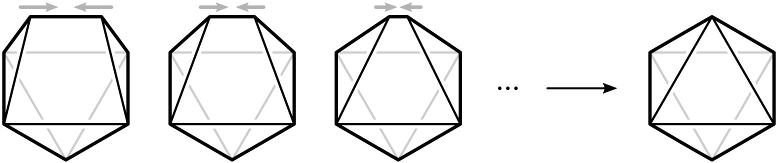
Induction base:

- $|E| = 6$ (simplex) is clearly rigid.

Induction step:

- Choose an edge $e \in E$ for which G_P/e is polyhedral.
- *Induction hypothesis*: there is a first-order rigid realizations P' of G_P/e.
- Choose a sequence of realizations $P_1, P_2, P_3, \ldots \longrightarrow P'$.
- Show: if each P_i has a non-zero stress, so does P'. $\lightning$

  $\longrightarrow$ some P_i must be first-order rigid.



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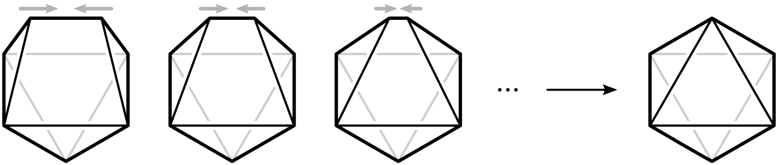

Stresses survive contraction

Given a sequence $P_1, P_2, P_3, \ldots \longrightarrow P'$ realizing the contracting $\hat{\imath}\hat{\jmath} \longrightarrow \hat{\hat{\jmath}}$.

Lemma.

If each P_n has a non-zero stress $(\boldsymbol{\omega}^n, \boldsymbol{\alpha}^n)$, then P' also has a non-zero stress $(\boldsymbol{\omega}', \boldsymbol{\alpha}')$.

$$\begin{aligned}
\omega_{ij}^n &\longrightarrow \omega'_{ij} && \text{if } i,j \notin \{\hat{\imath}, \hat{\jmath}\} \\
\omega_{i\hat{\imath}}^n + \omega_{i\hat{\jmath}}^n &\longrightarrow \omega'_{i\hat{\hat{\jmath}}} && \text{if } i \notin \{\hat{\imath}, \hat{\jmath}\} \\
\omega_{\hat{\imath}\hat{\jmath}}^n &\longrightarrow - \\
\alpha_{ik}^n &\longrightarrow \alpha'_{ik} && \text{if } i \notin \{\hat{\imath}, \hat{\jmath}\} \\
\alpha_{\hat{\imath}k}^n + \alpha_{\hat{\jmath}k}^n &\longrightarrow \alpha'_{\hat{\hat{\jmath}}k}
\end{aligned}$$

Stresses survive contraction

Given a sequence $P_1, P_2, P_3, \ldots \longrightarrow P'$ realizing the contracting $\hat\imath\hat\jmath \longrightarrow \hat{\hat{y}}$.

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If each P_n has a non-zero stress $(\boldsymbol{\omega}^n, \boldsymbol{\alpha}^n)$, then P' also has a non-zero stress $(\boldsymbol{\omega}', \boldsymbol{\alpha}')$.

$$
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\omega^n_{\hat\imath\hat\jmath} &\longrightarrow - && \\
\alpha^n_{ik} &\longrightarrow \alpha'_{ik} && \text{if } i \notin \{\hat\imath,\hat\jmath\} \\
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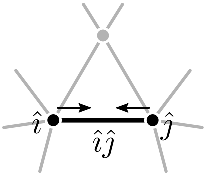
\end{aligned}
$$



Note: if F_k is a triangle, then $\alpha_{ik} = 0$.

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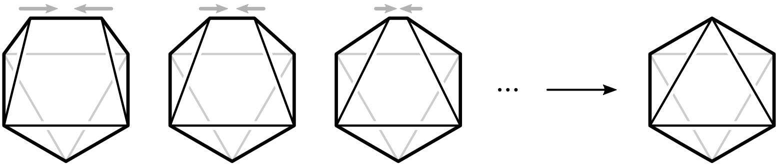
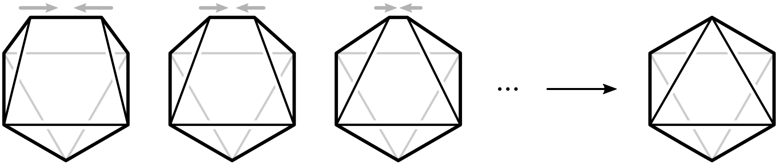


Contracting edges geometrically

How to find the sequence $P_1, P_2, P_3, \ldots \longrightarrow P'$'?



- Maxwell-Cremona correspondence

  "Polyhedral realizations are in 1:1 relation with planar stressed frameworks."

- Tutte embedding

  "One can prescribe the stresses of a planar framework."

Project and lift

Input: P'

Output: sequence $P_1, P_2, P_3, \ldots \longrightarrow P'$

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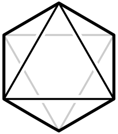

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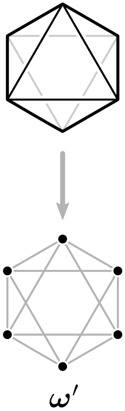

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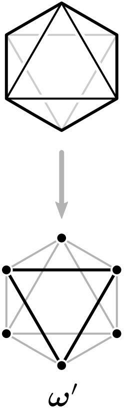

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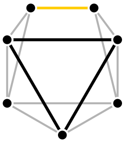
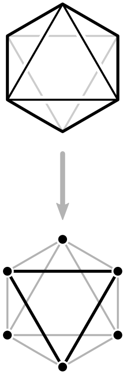


ω^1



ω'

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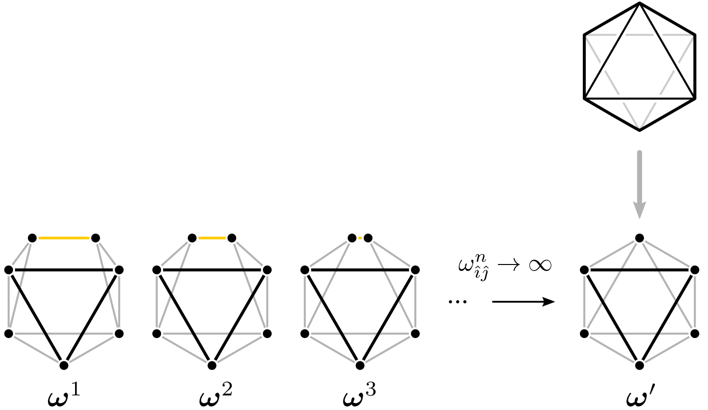

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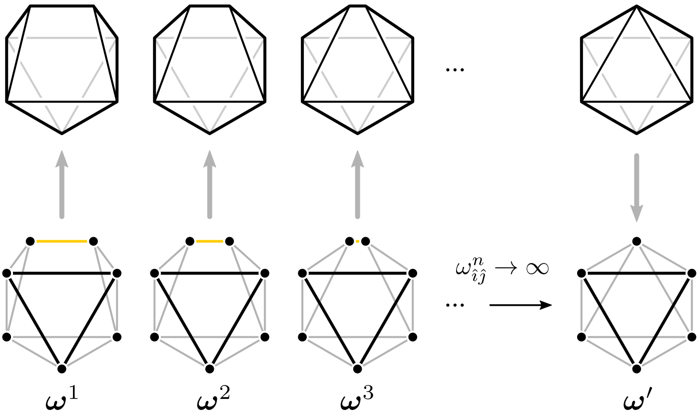

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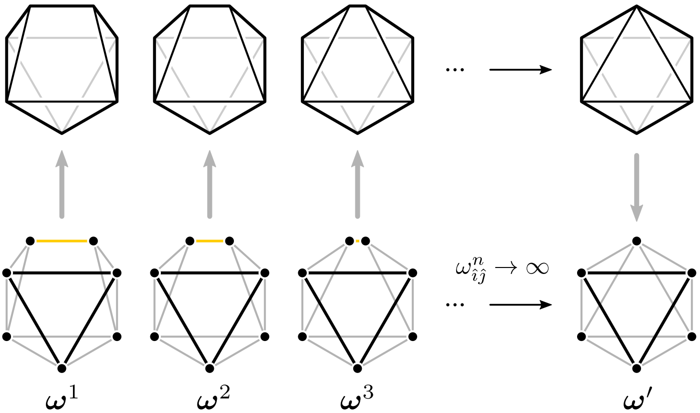

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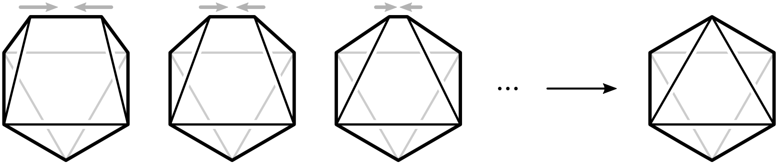

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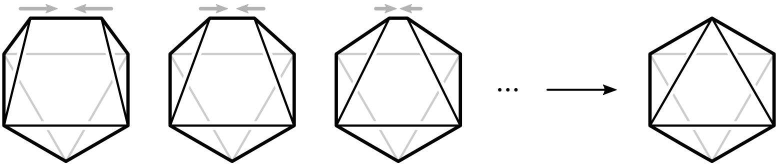

Many open questions

Higher dimensions

Question: Are polytopes of dimension $d \geq 4$ generically first-order rigid?

Problems:

- first-order rigid $\neq$ no non-zero stresses
- realization space is no longer contractible/connected/irreducible/...
- there are no useful analogues of Maxwell-Cremona/Tutte/...

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However ... can we pull the result up from dimension three?

- Fix a generic realization $P \subset \mathbb{R}^4$.

  $\longrightarrow$ all facets (= 3-dimensional faces) are generic.
- Suppose P has a first-order flex.

  $\longrightarrow$ induces a first-order flex on each facet.
- Since the facets are generic + 3D, the flexes on each facet must be trivial.
- Show: the flex of P must therefore be trivial as well. (Cauchy, Dehn)

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Beyond convexity

$$\textsc{Real}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{array} \;\middle|\; \langle p_i, n_k \rangle = 1 \;\text{ if } i \in F_k \right\}$$

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Question: Is $\textsc{Real}(\mathcal{P})$ irreducible?

or, alternatively,

Question: Can we run the "convex proof" once per irreducible component of $\textsc{Real}(\mathcal{P})$?

Thank you.