RANDOM TREES OF INTERMEDIATE VOLUME GROWTH

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(joint work with George Kontogeorgiou)

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Volume Growth in Graphs

$|B_v(r)|$
Volume growth

\[ B(v, r) := \{ x \in V(G) \mid \text{dist}(x, v) \leq r \} \]

\[ |B(v, 0)| = 1 \]
**Volume growth**

**ball** ... \( B(v, r) := \{ x \in V(G) \mid \text{dist}(x, v) \leq r \} \)

\[ |B(v, 1)| = 5 \]
**Volume growth**

**Ball** ... \( B(v, r) := \{ x \in V(G) \mid \text{dist}(x, v) \leq r \} \)

\[
|B(v, 2)| = 13
\]
**Volume growth**

**ball** ... \( B(v, r) := \{ x \in V(G) \mid \text{dist}(x, v) \leq r \} \)

\[ |B(v, 3)| = 25 \]
Examples: polynomial and exponential
**Geometric Group Theory**

Cayley graph of $\mathbb{Z}^2 = \langle x, y \mid xy = yx \rangle$. 

![Cayley graph of $\mathbb{Z}^2$](image-url)
**Geometric Group Theory**

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```
  o---o---o---o
  |   |   |   |
  o---o---o---o
  |   |   |   |
  o---o---o---o
  |   |   |   |
  o---o---o---o
```

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Geometric group theory

\[ \mathbb{Z} \]

\[ \mathbb{Z}^2 \]

\[ F_3 / \langle x^2, y^2, z^2 \rangle \]
Typical questions & results

Question: are there groups of intermediate growth?  
:= super-polynomial but sub-exponential  e.g. $\exp(r^{1/2})$ or $r^{\log r}$
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Yes: Grigorchuk group (1984)

$|B(e, r)| \sim \exp(r^\alpha)$ with $0.504 < \alpha < 0.7675$
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Theorem. (Gromov; 1981)

$G$ is of polynomial growth $\iff G$ is virtually nilpotent.

Theorem. (Trofimov; 1985)

Polynomial growth of vertex-transitive graphs must have integer degree.
### Beyond Cayley Graphs

<table>
<thead>
<tr>
<th>$B(v, r)$</th>
<th>$\sim r^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(v, r)$</td>
<td>$\in \theta(r^2)$</td>
</tr>
</tbody>
</table>
Volume Growth in Graphs

**Uniform growth**

Fix a function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

**Definition.**

A graph $G$ is of **uniform volume growth** $g$ if there are $c_1, c_2, C_1, C_2 > 0$ so that

$$C_1 \cdot g(c_1 r) \leq |B(v, r)| \leq C_2 \cdot g(c_2 r), \quad \text{for all } v \in V(G) \text{ and } r \geq 0.$$
Planar triangulations
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- Ambj et al. (1997); Angel (2003): planar triangulations of growth $\sim r^4$
  (quantum geometry)
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- **Benjamini, Schramm (2001):** triangulations from trees
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In the same paper: \( \sim r^\alpha \) for arbitrary \( \alpha \geq 1 \).
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- **Benjamini, Schramm (2001):** triangulations from trees

In the same paper: $\sim r^\alpha$ for arbitrary $\alpha \geq 1$.

- **Benjamini, Georgakopoulos (2021):** $\sim r^\alpha$ with $\alpha < 2$, then quasi-tree
Planar triangulations of growth $r^{3/2}$
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Trees
Uniform growth of trees

What kind of uniform growth can a tree have?

- linear ✓
- exponential ✓
Uniform growth of trees

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- linear ✓
- exponential ✓
- polynomial ??
- intermediate ??
- oscillating ??

(Benjamini, Schramm; 2001)

| $B(v,r)$ | $\sim r^{\alpha}$, where $\alpha = \log |E(T)|/\log \text{diam}(T) = \log 5/\log 3 \approx 1.464973$ |
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**The Question**

Q: “Are there unimodular trees of uniform intermediate volume growth?”

– Itai Benjamini

super-polynomial: $e^\omega(\log(r))$
sub-exponential: $e^{o(r)}$

↓
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**Idea:** find them as spanning trees of known intermediate growth graphs.
Trees
Trees
Does it work ... ?

Question

Given a graph of uniform growth $g$. Is there a (spanning) tree $T \subseteq G$ of the same uniform growth $g$?
Does it work ... ?

Question
Given a graph of uniform growth $g$. Is there a (spanning) tree $T \subseteq G$ of the same uniform growth $g$?

Turns out we don’t need the ambient graph!
The Construction

\[ T_0 \subset T_1 \subset T_2 \subset T_3 \subset \cdots \]
CONSTRUCTION – A SEQUENCE OF TREES

Given: sequence $\delta_1, \delta_2, \delta_3, ... \in \mathbb{N}$, $\delta_n \geq 1$

$\delta_n := n + 2345$

$T := \bigcup_n T_n$

$T_0$
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**Given:** sequence $\delta_1, \delta_2, \delta_3, \ldots \in \mathbb{N}$, $\delta_n \geq 1$

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Given: sequence $\delta_1, \delta_2, \delta_3, \ldots \in \mathbb{N}$, $\delta_n \geq 1$

$\delta_n := n + 2 \quad 3 \quad 4 \quad 5$

$T := \bigcup_{n} T_n$
HEURISTICS ARGUMENT

Properties of $T_n$:

- number of vertices: $(\delta_1 + 1) \cdots (\delta_n + 1)$
- distance from center to an apocentric vertex: $2^n - 1$
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The Construction

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$$|B(v, 2^n - 1)| = (\delta_1 + 1) \cdots (\delta_n + 1) \sim n! \sim (\log r)! \sim r^{\log \log r}$$

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**Heuristics Argument**

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- number of vertices: $(\delta_1 + 1) \cdots (\delta_n + 1)$
- distance from center to an apocentric vertex: $2^n - 1$

$$g(2^n - 1) \approx (\delta_1 + 1) \cdots (\delta_n + 1)$$
**Heuristics argument**

Properties of $T_n$:
- number of vertices: $(\delta_1 + 1) \cdots (\delta_n + 1)$
- distance from center to an apocentric vertex: $2^n - 1$

\[ g(2^n - 1) \approx (\delta_1 + 1) \cdots (\delta_n + 1) \quad \Rightarrow \quad \delta_n \approx \frac{g(2^n - 1)}{g(2^{n-1} - 1)} - 1 \]
**Example: Polynomial Growth**

\[ |B(v, r)| = (r + 1)^2 \]

\[ \delta_n := 3 \]
**Example: Polynomial Growth**

\[
\begin{align*}
|B(v, r)| &= (r + 1)^2 \\
|B(v, 2^n - 1)| &= (2^n)^2 = 4^n = (3 + 1) \cdots (3 + 1)
\end{align*}
\]

\[
\delta_n := 3
\]
**Example: Exponential Growth**

\[ \delta_n := 2^{2^n} \]

\[
|B(v, 2^n - 1)| = (\delta_1 + 1) \cdots (\delta_n + 1) = \prod_{k=1}^{n} \left( 2^{2^{k-1}} + 1 \right) = \sum_{i=0}^{2^n-1} 2^i = 2^{2^n} - 1 \\
\sim 2^{r+1} - 1
\]
Main Result

For every “nice” function $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ there is a tree of uniform growth $g$. 
What are “nice” functions?

- $g$ is increasing
- $g$ grows at least linearly
- $g$ grows at most exponentially
- $g$ does not oscillate between growth rates in unfortunate ways
**Main Result**

**What are “nice” functions?**

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- $g$ grows at least linearly ($\delta_n \geq 1$)
- $g$ grows at most exponentially
- $g$ does not oscillate between growth rates in unfortunate ways

\[ g(a + b) \geq g(a) + g(b) \]
\[ g(2^n + 1) \geq 2g(2^n) \]
\[ \delta_n \approx g(2^n + 1) - 1 \geq 1. \]
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- $g$ is increasing
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\[
g \text{ super-additive } \implies g(a + b) \geq g(a) + g(b) \\
\implies g(2^{n+1}) \geq 2g(2^n) \implies \delta_n \approx \frac{g(2^{n+1})}{g(2^n)} - 1 \geq 1.
\]
What are “nice” functions?

- $g$ is increasing
- $g$ grows at least linearly ($\delta_n \geq 1$)
- $g$ grows at most exponentially (bounded degree)
- $g$ does not oscillate between growth rates in unfortunate ways

\[ g \text{ super-additive } \implies g(a + b) \geq g(a) + g(b) \implies g(2^{n+1}) \geq 2g(2^n) \implies \delta_n \approx \frac{g(2^{n+1})}{g(2^n)} - 1 \geq 1. \]
Main Result: $T$ has uniform growth

$$\Delta(n) := \frac{\delta_n}{\delta_1 \cdots \delta_{n-1}}, \quad \bar{\Delta} := \sup_n [\Delta(n)], \quad \Gamma := \sup_{m \geq n} \left[ \frac{\Delta(m)}{\Delta(n)} \right].$$

Theorem. (Kontogeorgiou, W.; 2022)

For super-additive $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ exists a tree $T$ so that for all $v \in V(T)$ and $r \geq 0$

$$\left| B(v,r) \right| \geq C_1 \cdot g(r/4)$$

if $\bar{\Delta} < \infty$ then $\left| B(v,r) \right| \leq C_2 \cdot g(2r)^2$

if $\Gamma < \infty$ then $\left| B(v,r) \right| \leq C_3 \cdot g(4r)$

In particular, if $\Gamma < \infty$, then $T$ is of uniform growth $g.$
Main Result:

**Main result:** \( T \) has uniform growth

\[
\Delta(n) := \frac{\delta_n}{\delta_1 \cdots \delta_{n-1}}, \quad \bar{\Delta} := \sup_n \{\Delta(n)\}, \quad \Gamma := \sup_{m \geq n} \left[ \frac{\Delta(m)}{\Delta(n)} \right].
\]

**Theorem.** (Kontogeorgiou, W.; 2022)

For super-additive \( g : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \) exists a tree \( T \) so that for all \( v \in V(T) \) and \( r \geq 0 \)

\[
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if \( \bar{\Delta} < \infty \) then

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In particular, if \( \Gamma < \infty \), then \( T \) is of uniform growth \( g \).

**Theorem.**

If \( g \) is super-additive and (eventually) log-concave, then there is a tree of uniform volume growth \( g \).
Unimodular Trees
Q: “Are there unimodular trees of uniform intermediate volume growth?”

“unimodular random rooted trees”

– Itai Benjamini
Alternative limits

\[ T_0 \quad T_1 \quad T_2 \quad T_3 \]
Alternative limits
APOCENTRIC LIMIT
Benjamini-Schramm limits
Benjamini-Schramm limits
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Benjamini-Schramm limits
Benjamini-Schramm limits

$T_0, T_1, T_2, T_3, \ldots \xrightarrow{\text{BS}} \mathcal{T}$

- Benjamini-Schramm limits are unimodular
- A set of graphs of uniformly bounded degree is compact
- Every sequence of uniformly bounded degree has a convergent subsequence.

**Theorem.**

If $g$ is super-additive and (eventually) log-concave, then there is a unimodular random rooted tree of uniform volume growth $g$. 
A threshold phenomenon

Theorem. (structure theorem)

(i) if \( g \in \Omega(r^{\log \log r}) \), then \( \mathcal{T} \) is a.s. an apocentric limit.
(ii) if \( g \in \mathcal{O}(r^{\alpha \log \log r}) \) for some \( \alpha > 1 \), then \( \mathcal{T} \) is a.s. a mixed limit.

- if growth is fast enough the Benjamini-Schramm limit can be a deterministic tree.

\[
|B_T(v, r)| \sim \exp(r^\alpha) \quad \text{where } \alpha = \log(\phi) \approx 0.6942.
\]

Question

Do general unimodular trees of uniform growth show a similar threshold phenomenon?
Thank you.

G. Kontogeorgiou and M. Winter (2022), arXiv
“(Random) Trees of Intermediate Volume Growth”