(Random) trees of intermediate volume growth

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06. December, 2022

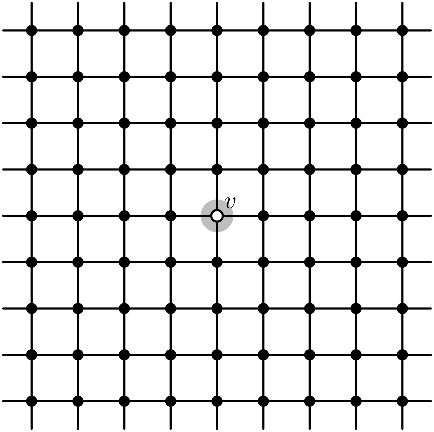
Volume Growth in Graphs

$$|B_v(r)|$$

Volume growth

ball ... $B(v,r) := \{\, x \in V(G) \mid \mathrm{dist}(x,v) \le r \,\}$

$|B(v,0)| = 1$



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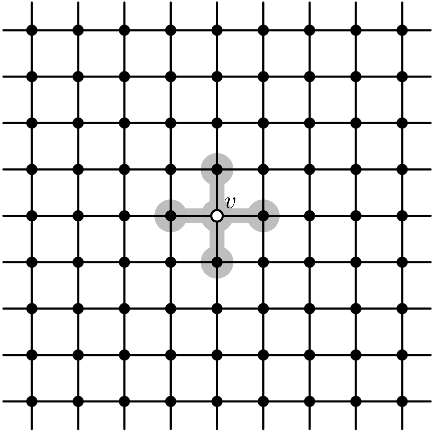
$|B(v,1)| = 5$



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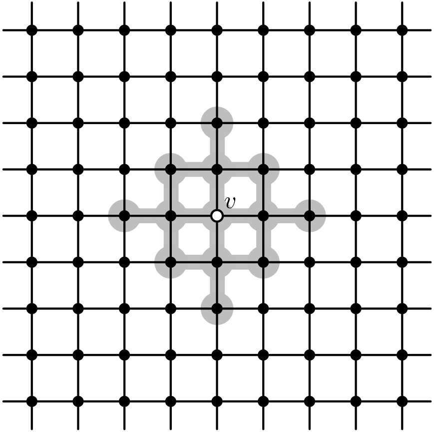
$|B(v,2)| = 13$



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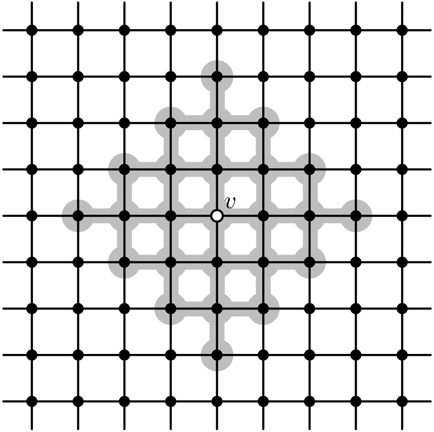
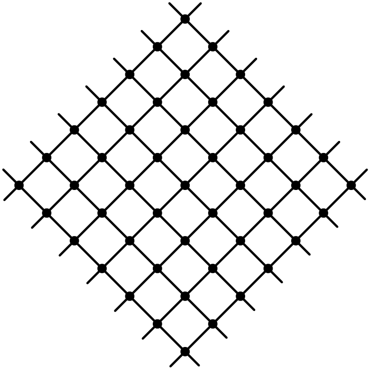
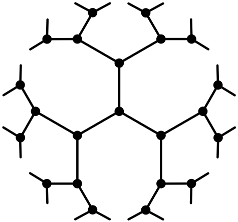
$|B(v,3)| = 25$



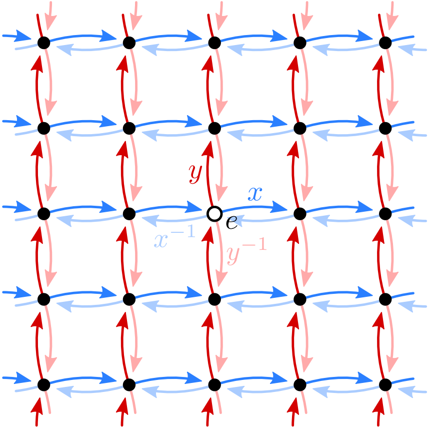
Examples: polynomial and exponential





Geometric group theory

Cayley graph of $\mathbb{Z}^2 = \langle x, y \mid xy = yx \rangle$.



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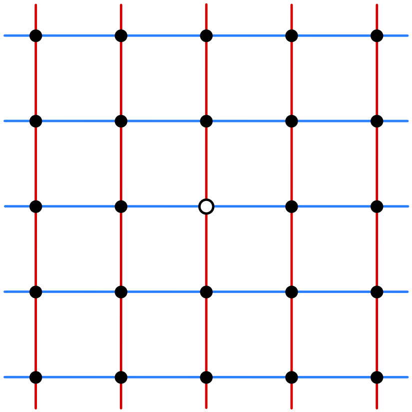

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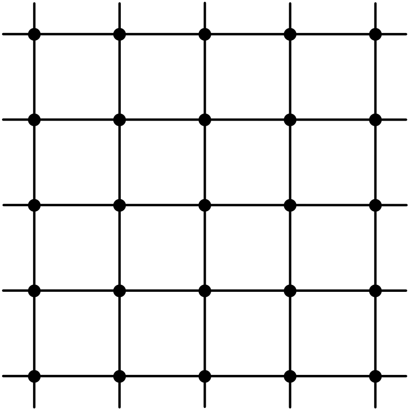
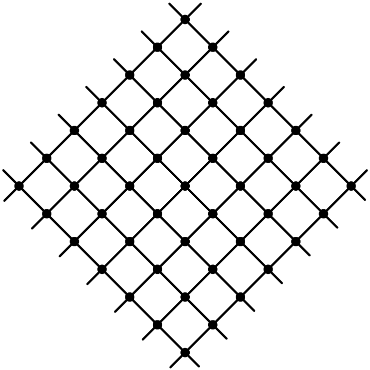
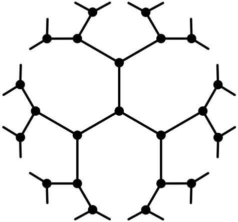


Geometric group theory

$\mathbb{Z}$



$\mathbb{Z}^2$



$F_3/\langle x^2, y^2, z^2\rangle$

Typical questions & results

Question: are there groups of *intermediate growth*?

$:=$ super-polynomial but sub-exponential e.g. $\exp(r^{1/2})$ or $r^{\log r}$

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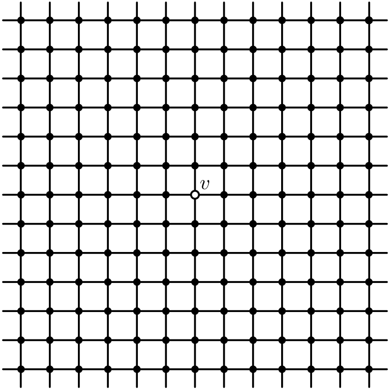
Theorem. (Gromov; 1981)

G is of polynomial growth $\Longleftrightarrow$ G is virtually nilpotent.

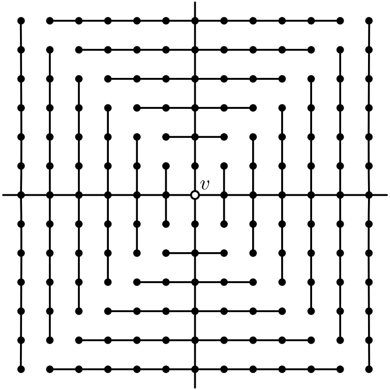
Theorem. (Trofimov; 1985)

Polynomial growth of vertex-transitive graphs must have integer degree.

Beyond Cayley graphs



$|B(v,r)| \sim r^2$



$|B(v,r)| \in \theta(r^2)$

Uniform growth

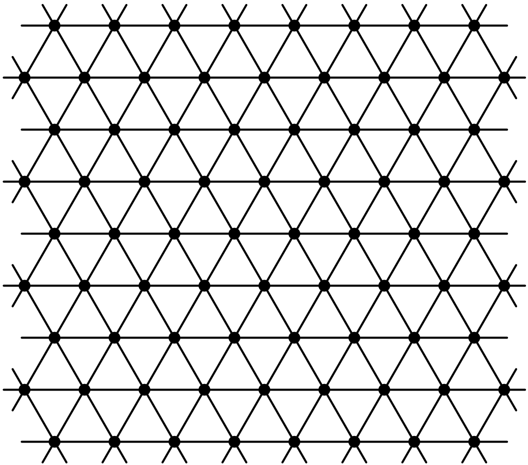
Fix a function $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

Definition.

A graph G is of **uniform volume growth** g if there are $c_1, c_2, C_1, C_2 > 0$ so that

$$C_1 \cdot g(c_1 r) \;\leq\; |B(v,r)| \;\leq\; C_2 \cdot g(c_2 r), \quad \text{for all } v \in V(G) \text{ and } r \geq 0.$$

Planar triangulations



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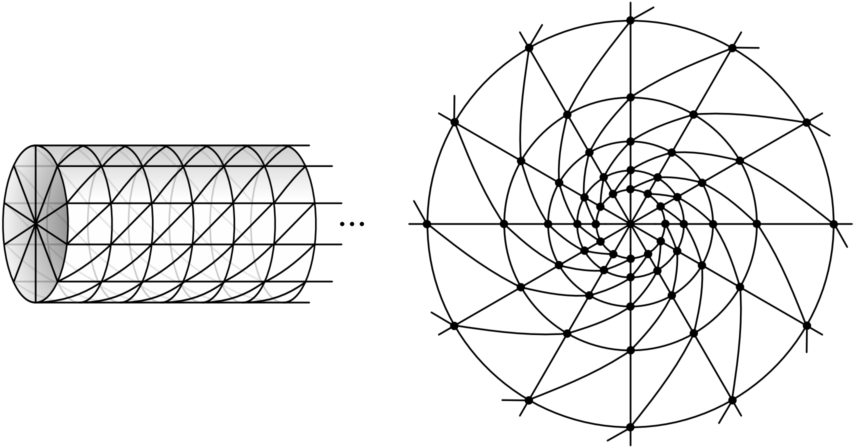

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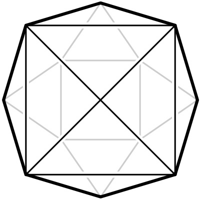

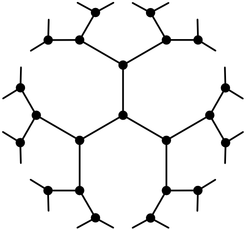

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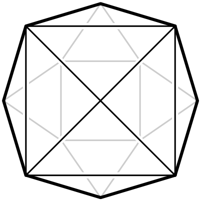

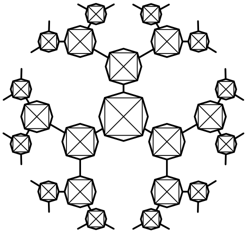

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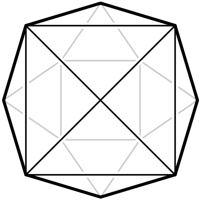

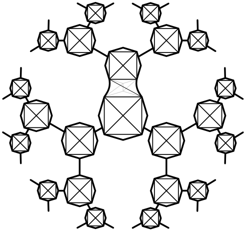

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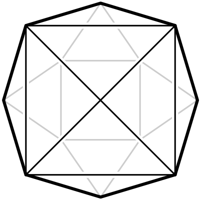

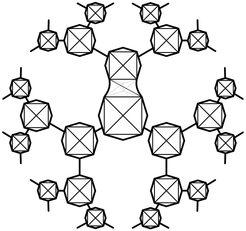

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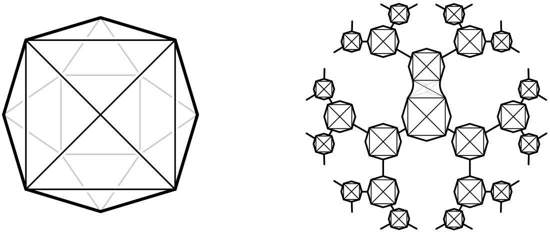
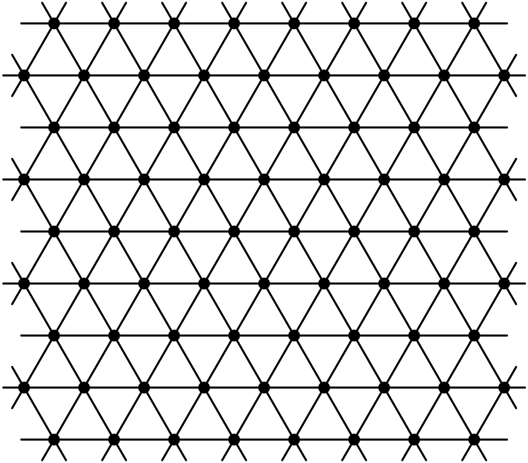


  In the same paper: $\sim r^\alpha$ for arbitrary $\alpha \geq 1$.
- Benjamini, Georgakopoulos (2021): $\sim r^\alpha$ with $\alpha < 2$, then quasi-tree

Planar triangulations of growth $r^{3/2}$



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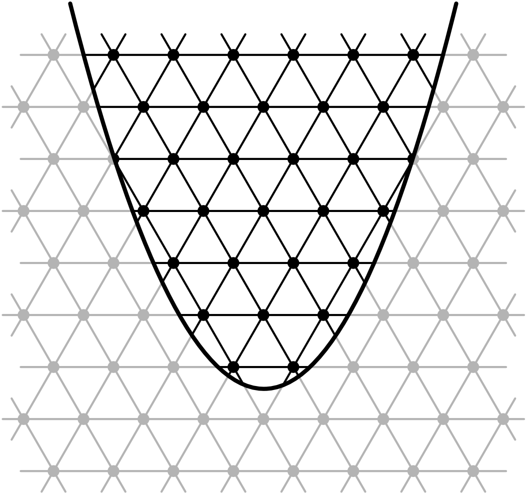

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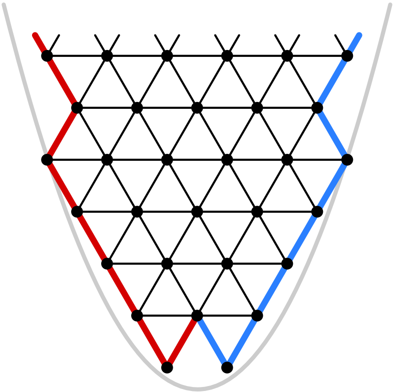

Trees

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- linear ✓
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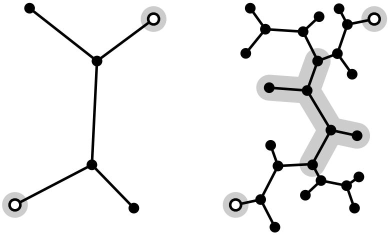
(Benjamini, Schramm; 2001)



$$|B(v,r)| \sim r^\alpha, \quad \text{where } \alpha = \frac{\log|E(T)|}{\log \operatorname{diam}(T)} = \frac{\log 5}{\log 3} \approx 1.464973.$$

The Question

super-polynomial: $e^{\omega(\log(r))}$
sub-exponential: $e^{o(r)}$
$\downarrow$

Q: *"Are there unimodular trees of uniform intermediate volume growth?"*

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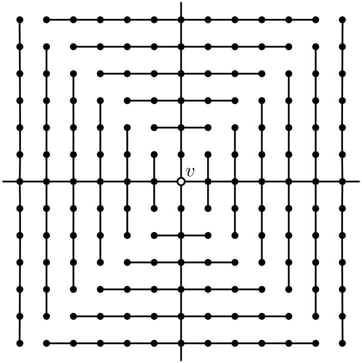
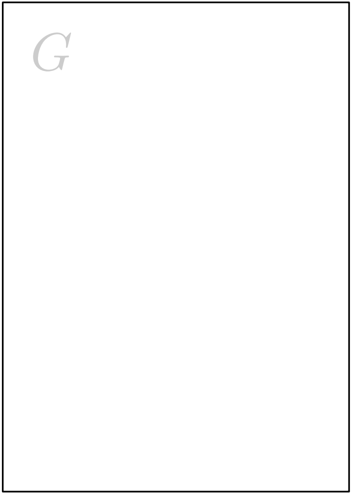
Idea: find them as spanning trees of known intermediate growth graphs.

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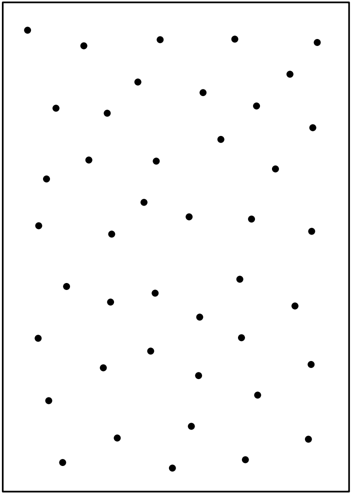

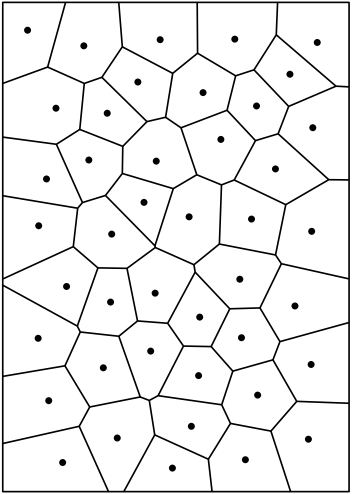

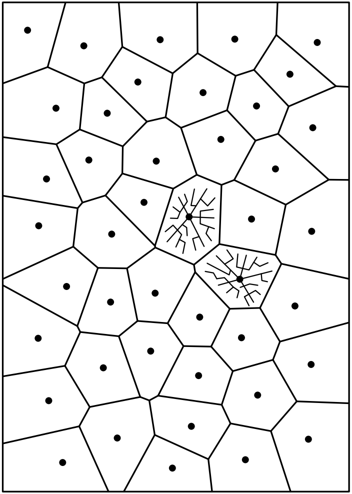

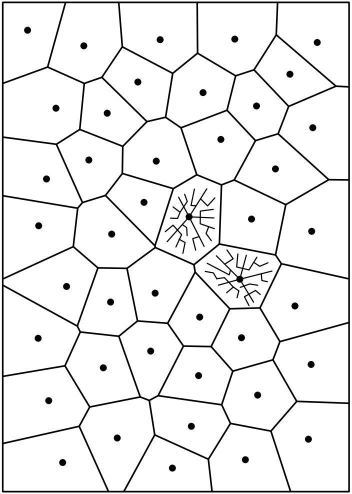

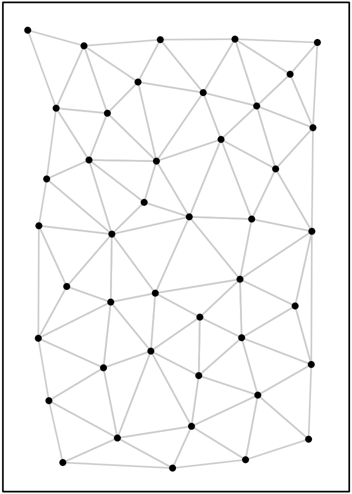

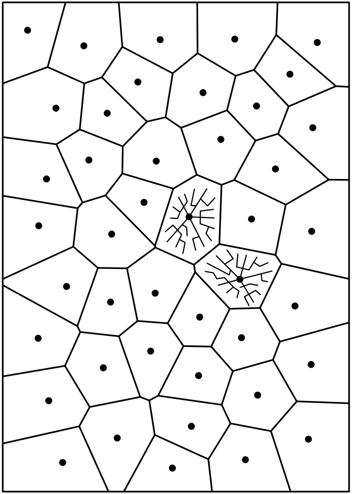

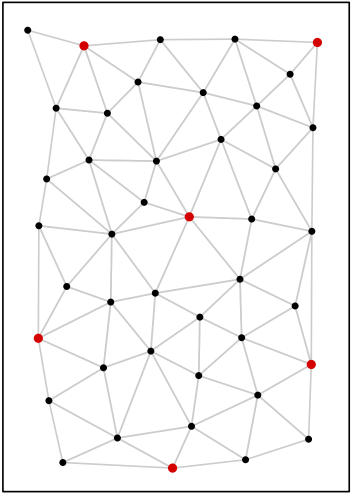

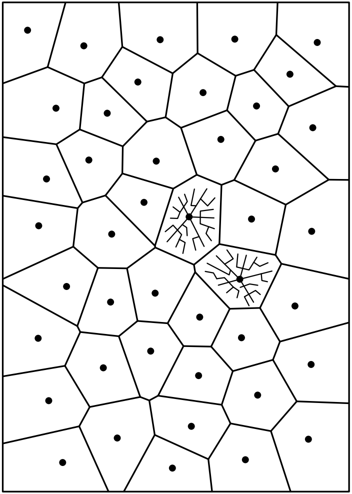

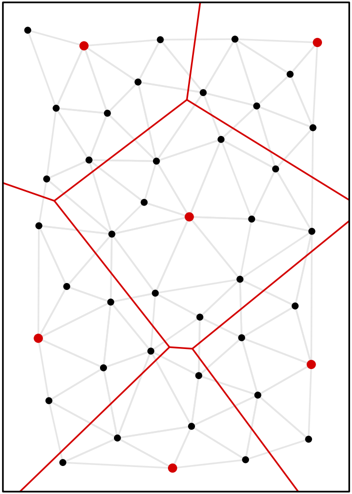

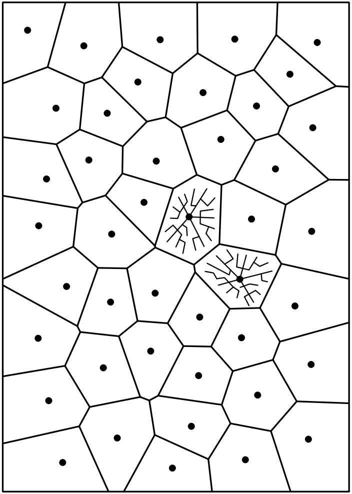

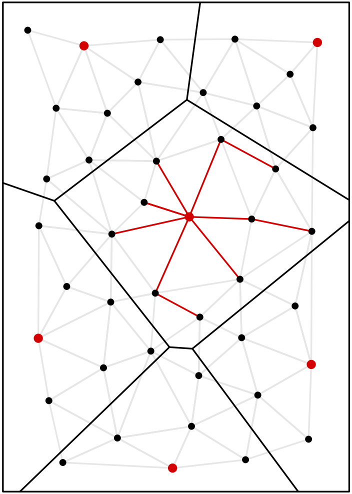

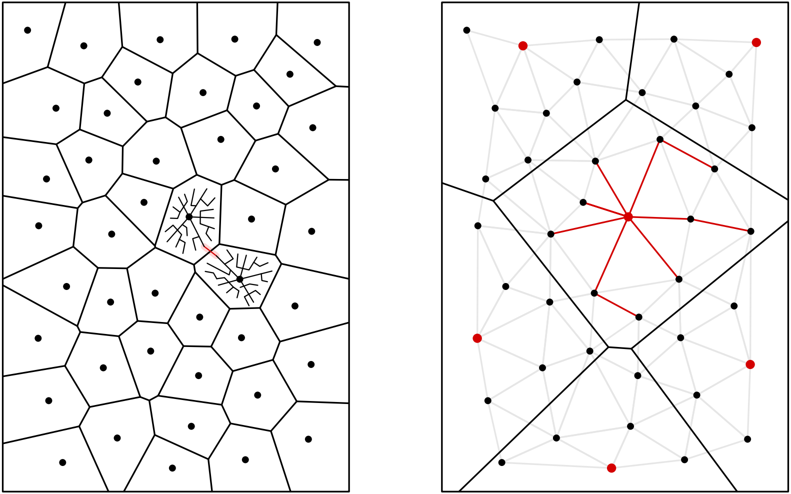

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Turns out we don't need the ambient graph!

The Construction

$$T_0 \subset T_1 \subset T_2 \subset T_3 \subset \cdots$$

Construction – a sequence of trees

Given: sequence $\delta_1, \delta_2, \delta_3, ... \in \mathbb{N},\ \delta_n \geq 1$

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$\delta_n := n + 2 \quad 3 \quad 4 \quad 5$

T_0

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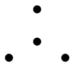

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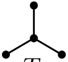

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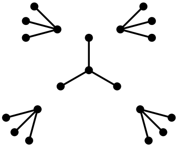

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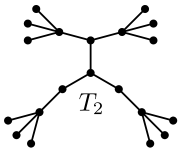

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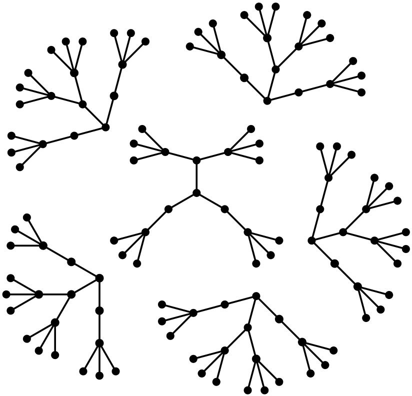

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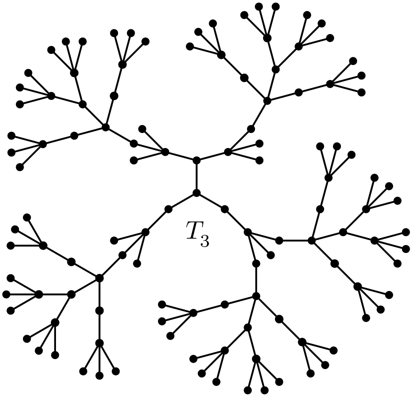

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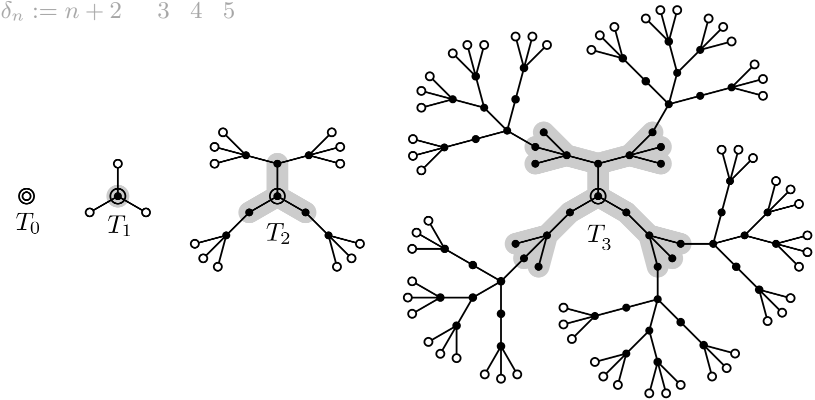

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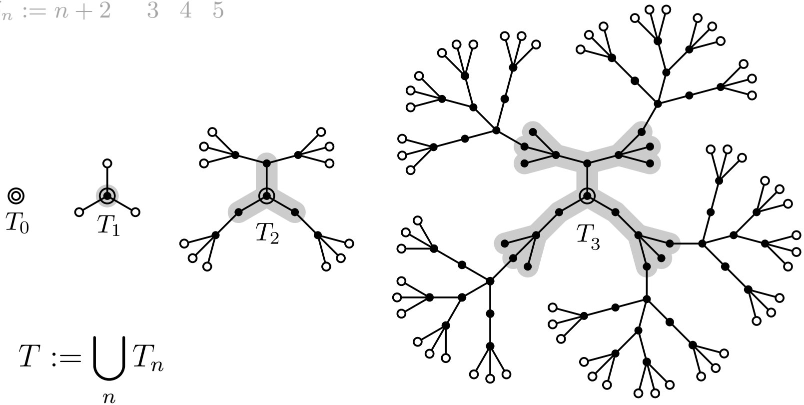
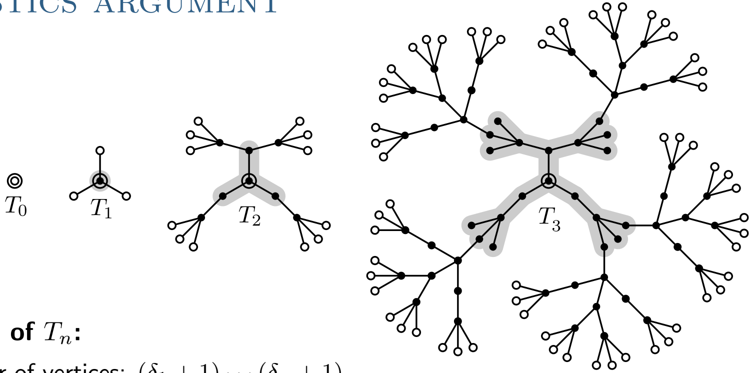


$$T := \bigcup_n T_n$$

Heuristics argument



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- number of vertices: $(\delta_1 + 1) \cdots (\delta_n + 1)$
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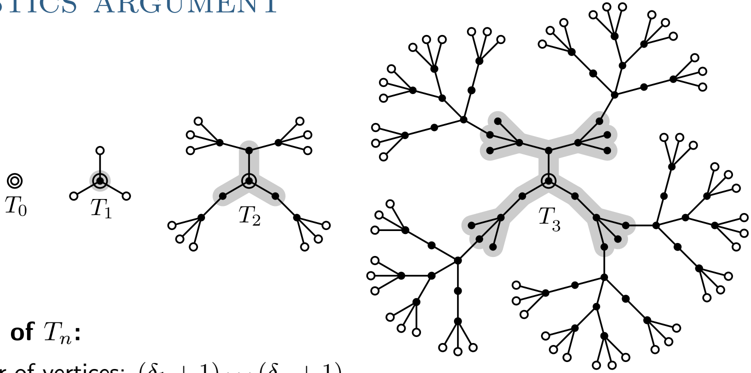

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$$|B(v, 2^n - 1)| = (\delta_1 + 1) \cdots (\delta_n + 1)$$

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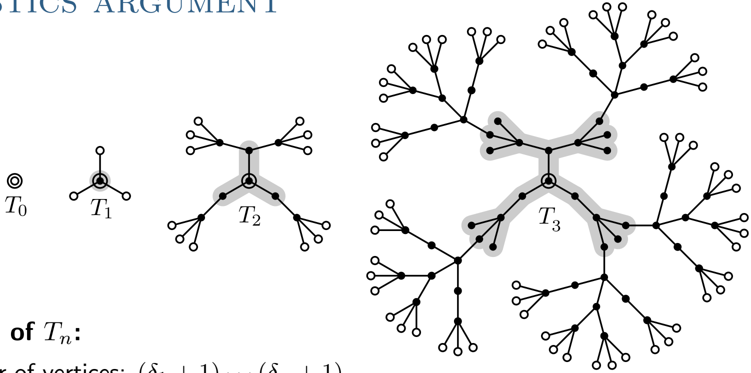

Properties of T_n:

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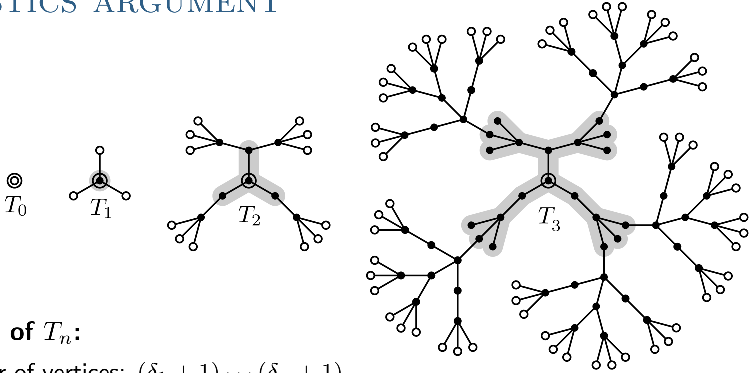

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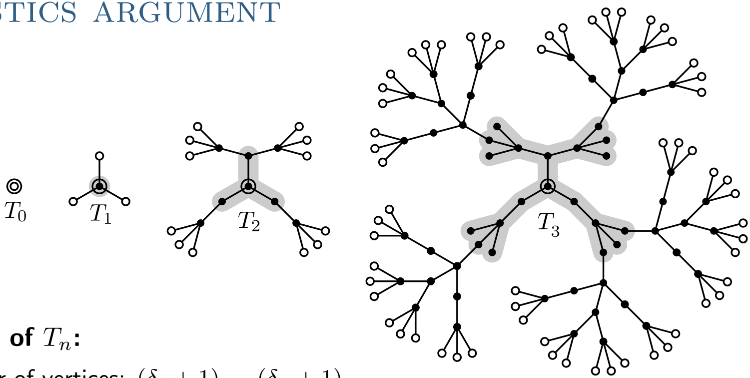

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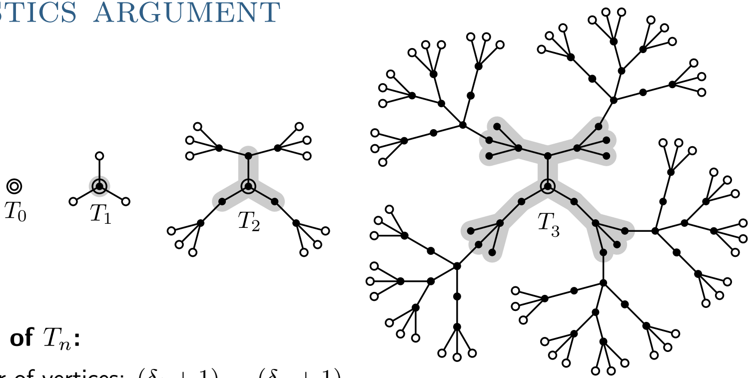

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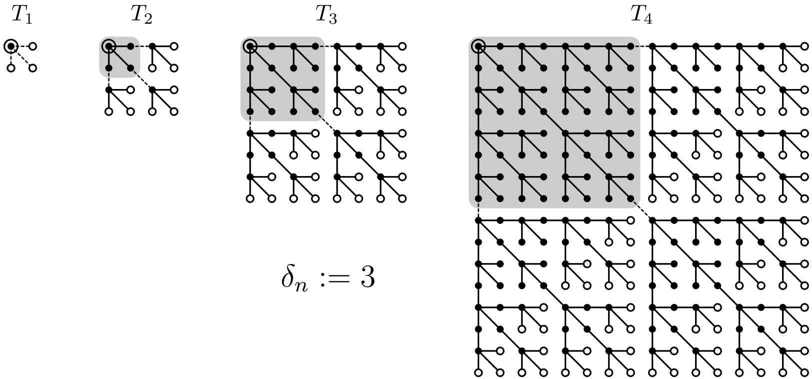
Example: polynomial growth

$$|B(v,r)| \overset{!}{=} (r+1)^2$$

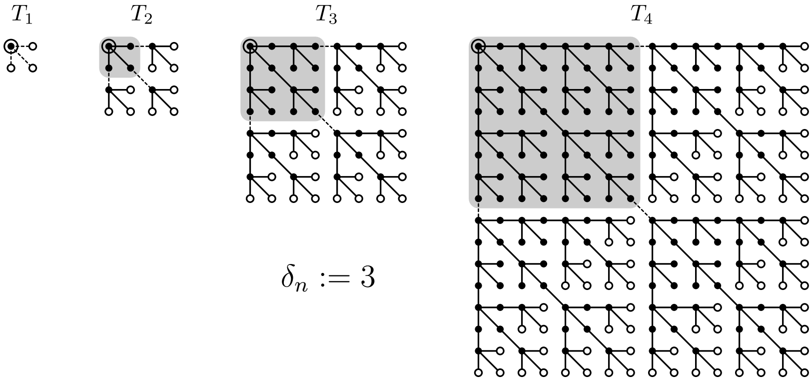


$$\delta_n := 3$$

Example: polynomial growth

$$(\delta_1+1)\cdots(\delta_n+1)$$

$$|B(v,r)| \stackrel{!}{=} (r+1)^2 \implies |B(v,2^n-1)| = (2^n)^2 = 4^n = (3+1)\cdots(3+1)$$



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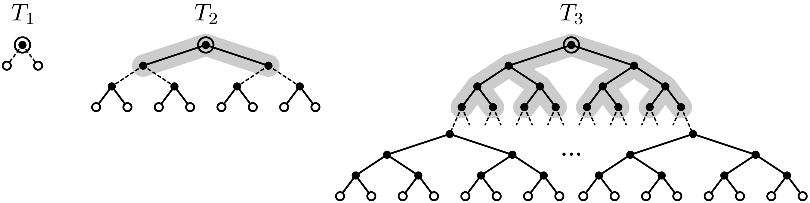
Example: exponential growth

$$\delta_n := 2^{2^n}$$



$$|B(v, 2^n - 1)| = (\delta_1 + 1) \cdots (\delta_n + 1) = \prod_{k=1}^{n} \left(2^{2^{k-1}} + 1\right) = \sum_{i=0}^{2^n - 1} 2^i = 2^{2^n} - 1$$

$$\sim 2^{r+1} - 1$$

Main Result

For every "nice" function $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ there is a tree of uniform growth g.

What are "nice" functions?

- g is increasing
- g grows at least linearly
- g grows at most exponentially
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Main result: T has uniform growth

max-degree − 1

$$\Delta(n) := \frac{\delta_n}{\delta_1 \cdots \delta_{n-1}}, \qquad \bar\Delta := \sup_n \lceil \Delta(n) \rceil, \qquad \Gamma := \sup_{m \ge n} \left\lceil \frac{\Delta(m)}{\Delta(n)} \right\rceil.$$

Theorem. (Kontogeorgiou, W.; 2022)

For super-additive $g\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ exists a tree T so that for all $v \in V(T)$ and $r \ge 0$

$$|B(v,r)| \ge C_1 \cdot g(r/4)$$

if $\bar\Delta < \infty$ then $$|B(v,r)| \le C_2 \cdot g(2r)^2$$

if $\Gamma < \infty$ then $$|B(v,r)| \le C_3 \cdot g(4r)$$

In particular, if $\Gamma < \infty$, then T is of uniform growth g.

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In particular, if $\Gamma < \infty$, then T is of uniform growth g.

Theorem.

If g is super-additive and (eventually) log-concave, then there is a tree of uniform volume growth g.

Unimodular Trees

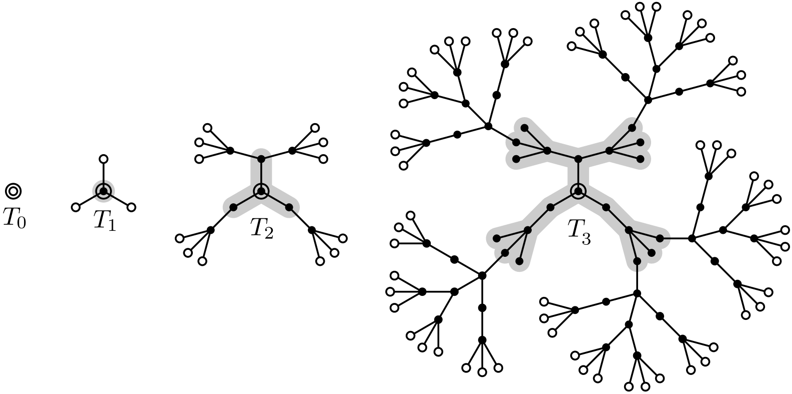
The original question

Q: *"Are there unimodular trees of uniform intermediate volume growth?"*

↑ *"unimodular random rooted trees"*

– Itai Benjamini

Alternative limits



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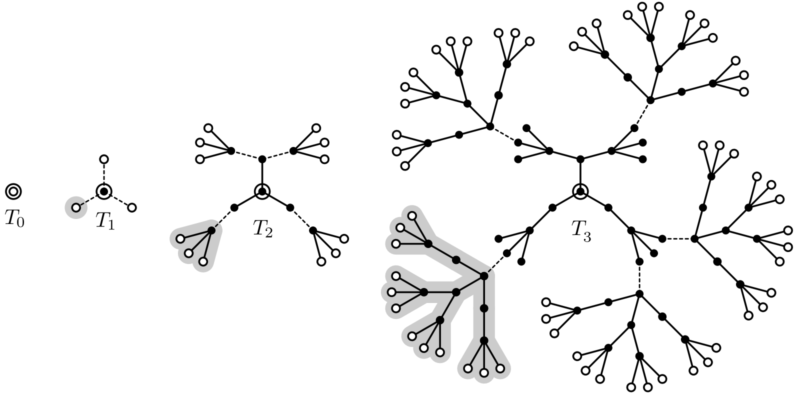
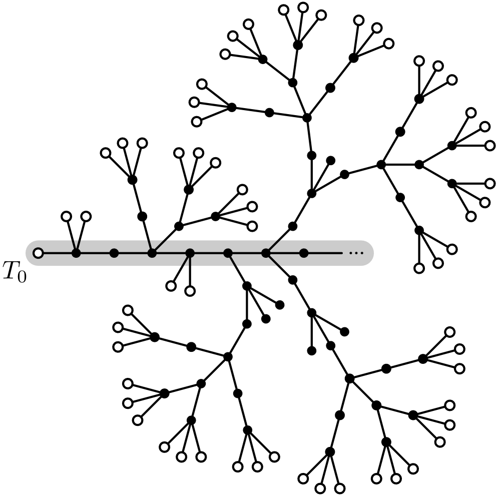
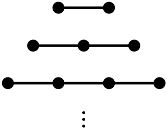


Apocentric limit



Benjamini-Schramm limits



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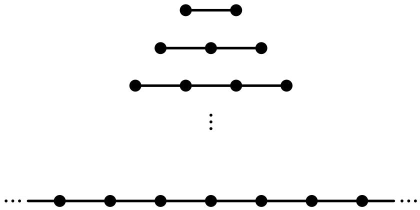

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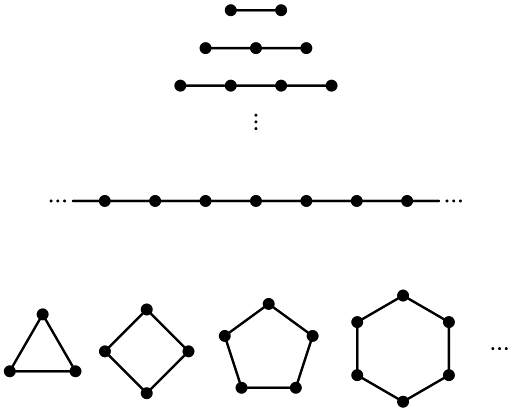

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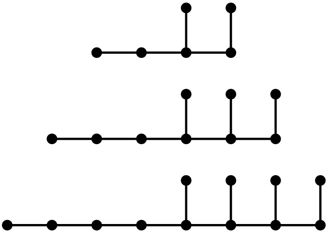

⋮

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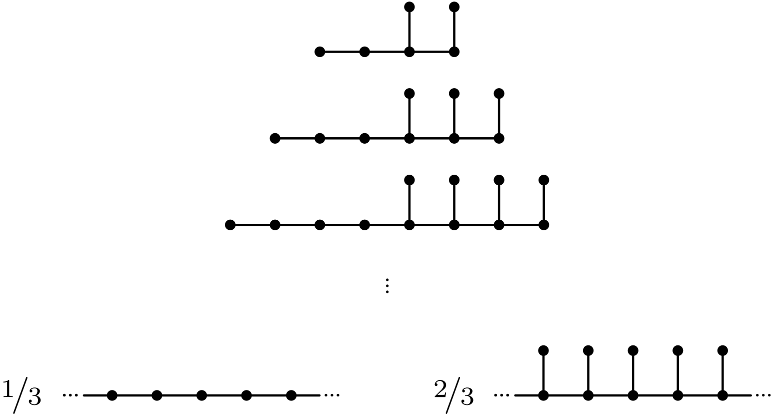


Benjamini-Schramm limits

$$T_0, T_1, T_2, T_3, \dots \quad \xrightarrow{\text{BS}} \quad \mathcal{T}$$

- Benjamini-Schramm limits are unimodular
- a set of graphs of uniformly bounded degree is *compact*
- every sequence of uniformly bounded degree has a convergent subsequence.

Theorem.

If g is super-additive and (eventually) log-concave, then there is a unimodular random rooted tree of uniform volume growth g.

A threshold phenomenon

Theorem. (structure theorem)

(i) *if* $g \in \Omega(r^{\log\log r})$, *then* $\mathcal{T}$ *is a.s. an apocentric limit.*

(ii) *if* $g \in \mathcal{O}(r^{\alpha \log\log r})$ *for some* $\alpha > 1$, *then* $\mathcal{T}$ *is a.s. a mixed limit.*

- if growth is fast enough the Benjamini-Schramm limit can be a deterministic tree.

$$|B_T(v,r)| \sim \exp(r^\alpha) \quad \text{where } \alpha = \log(\phi) \approx 0.6942.$$

Question

Do general unimodular trees of uniform growth show a similar threshold phenomenon?

Google

Trees of intermediate growth

Ähnliche Fragen

What are the stages of tree growth?

What is the growth of a tree called?

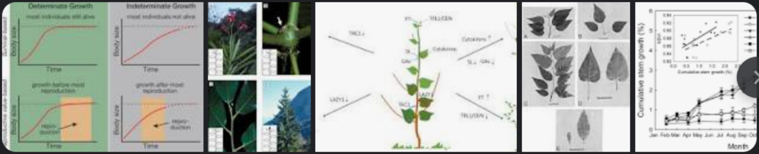
Why do plants have indeterminate growth?

What might different species of trees in a forest compete for?

Feedback geben

Bilder zu Trees of intermediate growth

apical meristem | tree trunk | canopy closure | tree ring



Thank you.

G. Kontogeorgiou and M. Winter (2022), arXiv
"(Random) Trees of Intermediate Volume Growth"