Infinite linkless graphs and their forbidden minors

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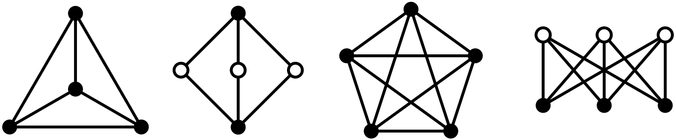
Minor closed families

Example: planar graphs

planar := can be drawn in $\mathbb{R}^2$ without crossing edges



K_4 ✓ $K_{2,3}$ ✓ K_5 ✗ $K_{3,3}$ ✗

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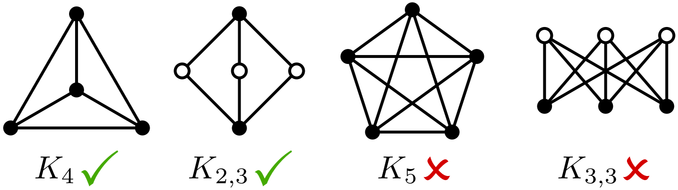
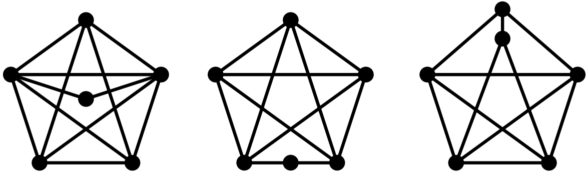


Theorem. (Kuratowski, 1930)

A graph is planar if and only if it "contains" no K_5 or $K_{3,3}$.

There are only finitely many finite reasons to be non-planar.

Forbidden minors



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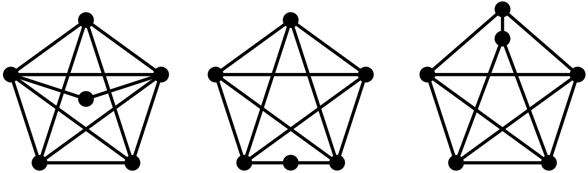

Minor $:=$ obtained by repeated edge deletion and contraction

Theorem. (Kuratowski, 1930)

A graph is planar if and only if it contains no K_5- or $K_{3,3}$-minor.

$=:$ *"forbidden minor characterization" of planarity*

Other topological graph classes

linkless := can be embedded into $\mathbb{R}^3$ without linking cycles

knotless := can be embedded into $\mathbb{R}^3$ without knotted cycles

flat := can be embedded into $\mathbb{R}^3$ so that every cycle can be filled in by a disc

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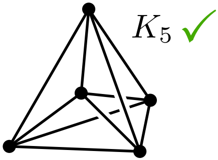
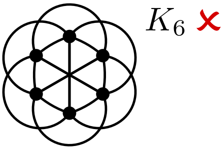
Theorem. (Robertson, Seymour, Thomas, 1993)

G is linkless $\iff$ G is flat

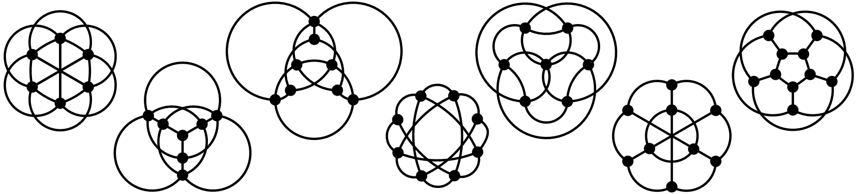




Robertson-Seymour Theorem

Theorem. (Robertson, Seymour, 1983-2004)

Every minor-closed family of finite graphs has a finite forbidden minor characterization (i.e. is characterized by finitely many forbidden minors, each of which is finite)



Infinite graphs

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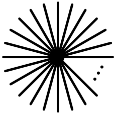
$$\big(\,\mathbb{R}^2,\, x \sim y \iff \|x-y\| = 1\,\big)$$



For our purpose, you might think of graphs that are

- countable
- locally finite

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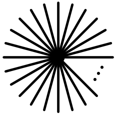

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The *Robertson-Seymour theorem* does not apply to infinite graphs.

Still we hope ...

An (infinite) graph is X, if and only if every finite subgraph is X.

$\implies$ same forbidden minors as finite graphs

Example: infinite planar graphs

Theorem. (Erdős)

An (infinite) graph G is planar, if and only if every finite subgraph is planar.

$\Longrightarrow$ infinite planar graphs have no K_5- and $K_{3,3}$-minors

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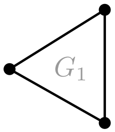

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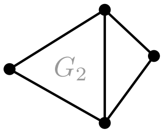

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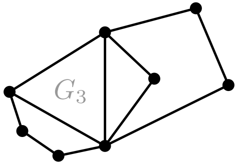

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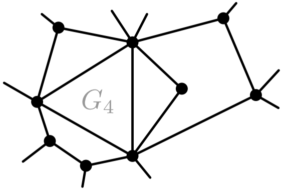

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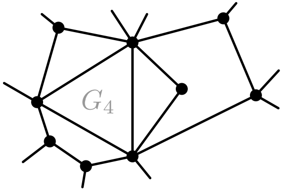

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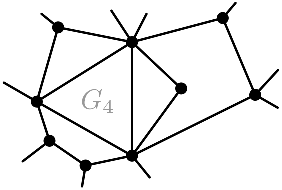

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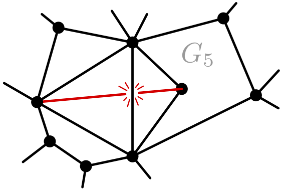

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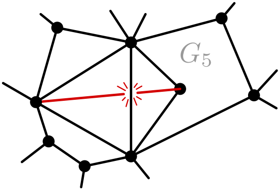

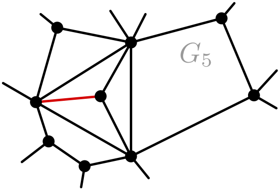

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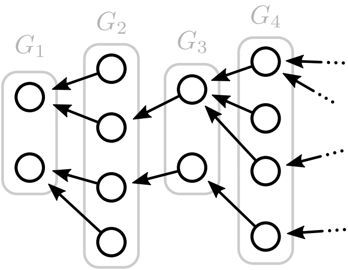
Proof. (compactness argument)



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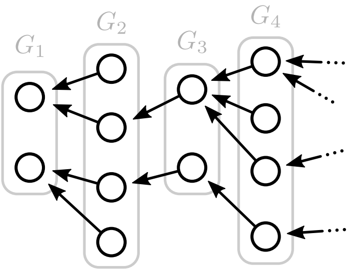

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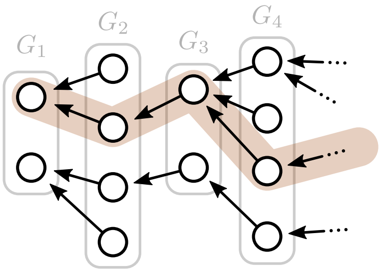
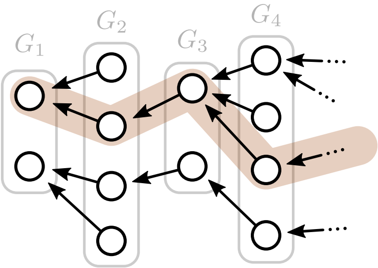


Beyond planar graphs



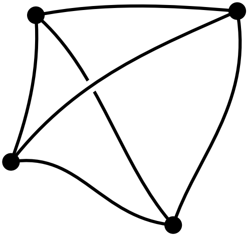
Question: Can we do the same for other graphs classes?

We need ... (an equivalent of Whitney's theorem)

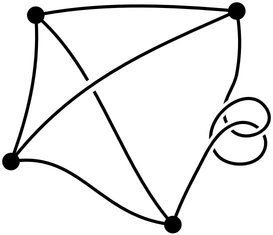
... if G can be embedded with property X, then there are there only finitely many ways to do so (up to ambient isotopy).

Infinite linkless & flat graphs

How many linkless embeddings?



How many linkless embeddings? (∞ many)



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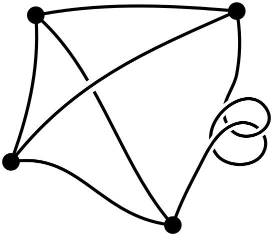
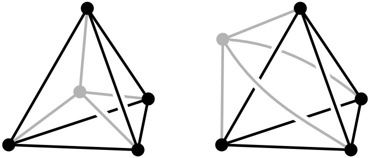


But: this is *not* a flat embedding !

Finitely many flat embeddings

Theorem. (Robertson, Seymour, Thomas, 1993)

- *K_5 and $K_{3,3}$ have exactly two flat embeddings.* (up to ambient isotopy)
- *Different flat embeddings of G differ in a K_5- or $K_{3,3}$-minor.*



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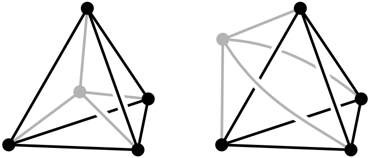

Corollary.

If G is finite and linkless and its number of K_5- and $K_{3,3}$-minors is N, then it has at most 2^N different flat embeddings.

Characterizing infinite linkless graphs

Theorem. (Kontogeorgiou, W., 2023+)

A graph is linkless if and only if every finite subgraph is linkless.

Corollary.

(Infinite) linkless graphs are characterized by the Petersen minors.

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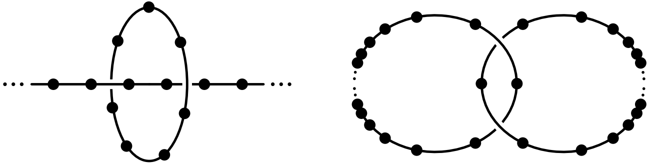
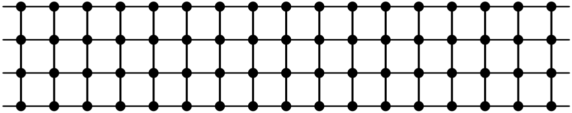
- there could be "infinite linked cycles"



Ends & the Freudenthal compactification

end := equivalence class of infinite rays that "go in the same direction"



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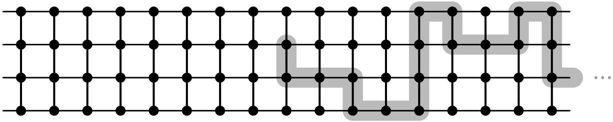

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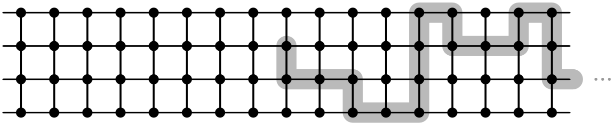
Freudenthal compactification

$:=$ topological space that contains G and a new point for each end



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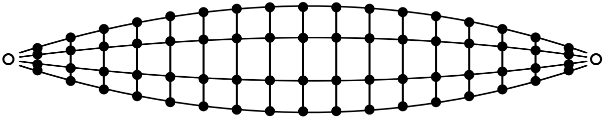

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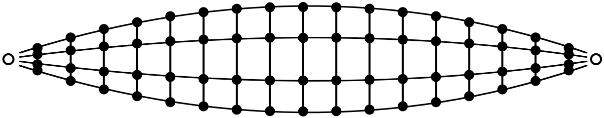

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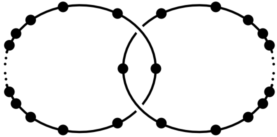
- How to construct a Freudenthal embedding?
- Is it still linkless?
- Can it be made flat?



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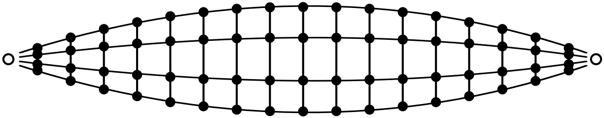

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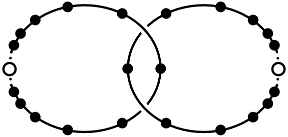

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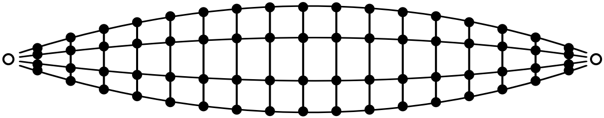

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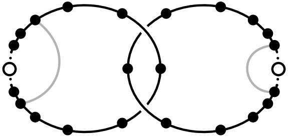

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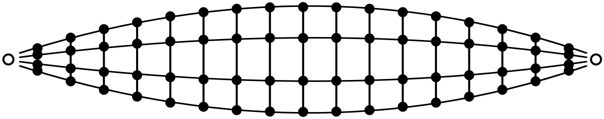

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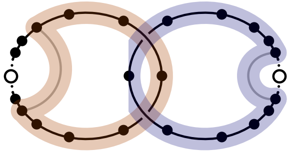


Decompositions and good graphs

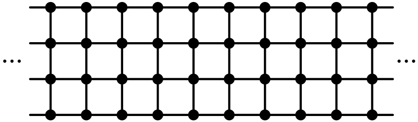
Definition.

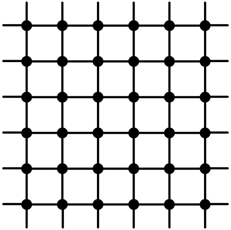
A graph is **good** if it has a decomposition $V(G) = S_1 \mathbin{\dot\cup} S_2 \mathbin{\dot\cup} \cdots$ of its vertex set satisfying the following:

- the induced subgraphs $G[S_i]$ are finite and connected,
- contracting the subgraphs $G[S_i]$ yields a forest.

Examples: locally finite graphs

Counterexample: the infinite clique





Decompositions and good graphs

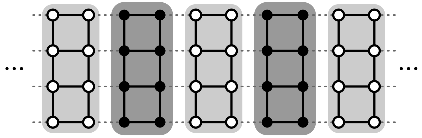
Definition.

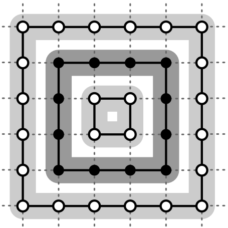
A graph is **good** if it has a decomposition $V(G) = S_1 \mathbin{\dot\cup} S_2 \mathbin{\dot\cup} \cdots$ of its vertex set satisfying the following:

- the induced subgraphs $G[S_i]$ are finite and connected,
- contracting the subgraphs $G[S_i]$ yields a forest.

Examples: locally finite graphs

Counterexample: the infinite clique

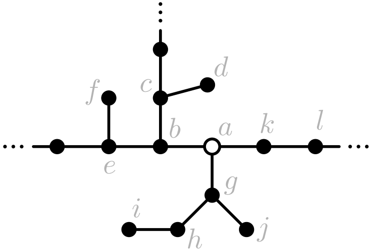


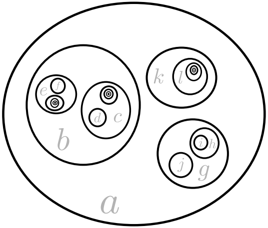


Linkless Freundenthal embeddings

Theorem. (Kontogeorgiou, W., 2023+)

A good linkless graph has a linkless Freudenthal embedding.





Infinite flat embeddings

Theorem. (Kontogeorgiou, W., 2023+)

A good graph is flat if and only if every finite subgraph is flat.

Corollary.

Also for infinite graphs holds: linkless $\Longleftrightarrow$ *flat.*

Infinite flat embeddings

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	linkless	flat
finite cycles only	✓	✓
infinite cycles included	✓	???

Question: has a (good) flat graph a flat Freudenthal embedding?

Thank you.