On 4-flat graphs and the embeddability of 2-complexes in dimension four

Martin Winter
this is joint work with Agelos Georgakopoulos and Tam Nguyen-Phan

University of Warwick

12. January, 2023

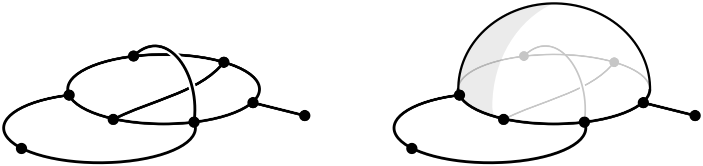
CW-complexes (cellular complexes)

Each **CW-complexes** C in this talk is 2-dimensional and composed of

- vertices (0-cells)
- edges (1-cells)

} 1-skeleton $C^{(1)}$ = a topological graph

- 2-cells ← 2-discs glued to the 1-skeleton



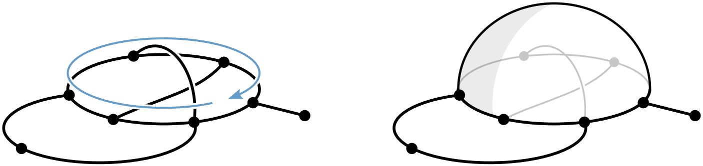
CW-complexes (cellular complexes)

Each **CW-complexes** C in this talk is 2-dimensional and composed of

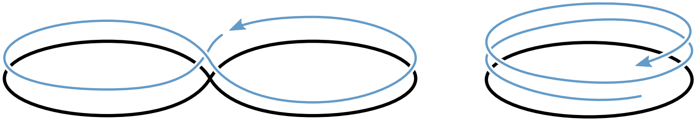
- vertices (0-cells)
- edges (1-cells)
- 2-cells

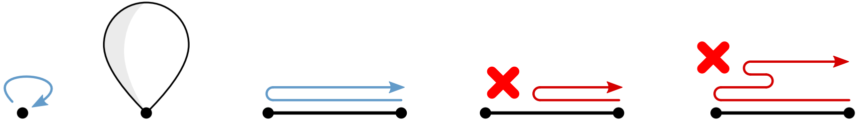
} 1-skeleton $C^{(1)}$ = a topological graph (covers vertices and edges)

← 2-discs glued to the 1-skeleton (for 2-cells)



2-cell attachment (simplicial, not necessarily injective)





Embedding complexes

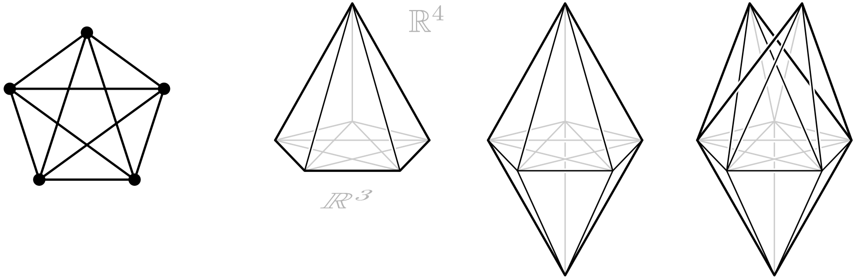
$$\phi\colon C \to \mathbb{R}^d$$

All embeddings in this talk are **piecewise linear** (PL).

- every d-complex embeds in $\mathbb{R}^{2d+1}$
- some d-complexes embed in $\mathbb{R}^{2d}$ (e.g. manifolds), others do not
  $\longrightarrow$ van Kampen obstruction
- this obstruction is “if and only if”, except for embeddings 2D $\to$ 4D
  (codimension 2 is the worst)

Non-embeddable in $\mathbb{R}^4$

Example: triple cone over K_5



Can I tell embeddability from the graph?

Observation: there are graphs, no matter what 2-cells we attach, the 2-complex always embeds in $\mathbb{R}^4$.

Example: trees, cycles

What about $K_4, K_5, K_6, K_7, \ldots$?

van der Holst's
4-flat graphs

Full complexes and 4-flat graphs

Definition.

Let G be a graph.

- the **full regular complex** $\mathcal{C}_{\text{reg}}(G)$ is the 2-complex obtained from G by attaching a 2-cell along each cycle of G.
- G is **4-flat** if $\mathcal{C}_{\text{reg}}(G)$ embeds in $\mathbb{R}^4$.

Full complexes and 4-flat graphs

Definition.

Let G be a graph.

- the **full regular complex** $\mathcal{C}_{\mathrm{reg}}(G)$ is the 2-complex obtained from G by attaching a 2-cell along each cycle of G.
- G is **4-flat** if $\mathcal{C}_{\mathrm{reg}}(G)$ embeds in $\mathbb{R}^4$.

- ... along each closed walk $\longrightarrow$ $\mathcal{C}(G)$ $\longrightarrow$ strongly 4-flat
- ... along each cycle $\longrightarrow$ $\mathcal{C}_{\mathrm{reg}}(G)$ $\longrightarrow$ 4-flat
- ... along each induced cycle $\longrightarrow$ $\mathcal{C}_{\mathrm{ind}}(G)$ $\longrightarrow$ weakly 4-flat

Full complexes and 4-flat graphs

Definition.

Let G be a graph.

- the **full regular complex** $\mathcal{C}_{\mathrm{reg}}(G)$ is the 2-complex obtained from G by attaching a 2-cell along each cycle of G.
- G is **4-flat** if $\mathcal{C}_{\mathrm{reg}}(G)$ embeds in $\mathbb{R}^4$.

- ... along each closed walk $\longrightarrow$ $\mathcal{C}(G)$ $\longrightarrow$ strongly 4-flat
- ... along each cycle $\longrightarrow$ $\mathcal{C}_{\mathrm{reg}}(G)$ $\longrightarrow$ 4-flat
- ... along each induced cycle $\longrightarrow$ $\mathcal{C}_{\mathrm{ind}}(G)$ $\longrightarrow$ weakly 4-flat

Theorem. (Georgakopoulos, Nguyen-Phan, W., 2023+)

strongly 4-flat $\Longleftrightarrow$ *4-flat* $\Longleftrightarrow$ *weakly 4-flat*

Example: K_6

How to embed $\mathcal{C}_{\text{ind}}(K_6)$?

- the only induced cycles of K_6 are triangles
- K_6 is the edge-graph of the 5-simplex
- the 5-simplex contains a 2-face (i.e. a 2-cell) in each triangle
- the Schlegel diagram of the 5-simplex in $\mathbb{R}^4$ contains an embedding of $\mathcal{C}_{\text{ind}}(K_6)$

Example: K_6 AND $K_7 - e$

How to embed $\mathcal{C}_{\text{ind}}(K_6)$?

- the only induced cycles of K_6 are triangles
- K_6 is the edge-graph of the 5-simplex
- the 5-simplex contains a 2-face (i.e. a 2-cell) in each triangle
- the Schlegel diagram of the 5-simplex in $\mathbb{R}^4$ contains an embedding of $\mathcal{C}_{\text{ind}}(K_6)$

How to embed $\mathcal{C}_{\text{ind}}(K_7 - e)$?

- ...
- $K_7 - e$ is the edge-graph of two 5-simplices glued at a facet
- ...

Example: K_6 AND $K_7 - e$

How to embed $\mathcal{C}_{\text{ind}}(K_6)$?

- the only induced cycles of K_6 are triangles
- K_6 is the edge-graph of the 5-simplex
- the 5-simplex contains a 2-face (i.e. a 2-cell) in each triangle
- the Schlegel diagram of the 5-simplex in $\mathbb{R}^4$ contains an embedding of $\mathcal{C}_{\text{ind}}(K_6)$

How to embed $\mathcal{C}_{\text{ind}}(K_7 - e)$?

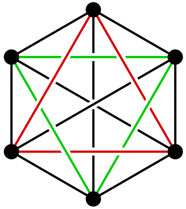
- ...
- $K_7 - e$ is the edge-graph of two 5-simplices glued at a facet
- ...

K_7 is <u>not</u> 4-flat ... by van Kampen obstruction.

Facts about 4-flat graphs

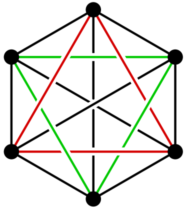
- planar graphs are 4-flat

Facts about 4-flat graphs



- planar graphs are 4-flat
- linkless graphs (aka. flat graphs) are 4-flat

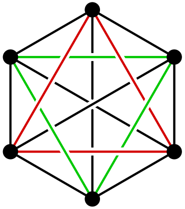
Facts about 4-flat graphs



- planar graphs are 4-flat
- linkless graphs (aka. flat graphs) are 4-flat
- cones over linkless graphs are 4-flat (this includes K_6)

$\rightarrow$ 4-flat graphs feel like a natural continuation of planar, linkless, ...

Facts about 4-flat graphs



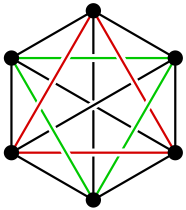
- planar graphs are 4-flat
- linkless graphs (aka. flat graphs) are 4-flat
- cones over linkless graphs are 4-flat (this includes K_6)

  $\rightarrow$ 4-flat graphs feel like a natural continuation of planar, linkless, ...

- 4-flat graphs form a **minor-closed family**

  $\rightarrow$ characterized by forbidden minors (Robertson–Seymour theorem)

Facts about 4-flat graphs



- planar graphs are 4-flat
- linkless graphs (aka. flat graphs) are 4-flat
- cones over linkless graphs are 4-flat (this includes K_6)

  $\rightarrow$ 4-flat graphs feel like a natural continuation of planar, linkless, ...

- 4-flat graphs form a **minor-closed family**

  $\rightarrow$ characterized by forbidden minors (Robertson–Seymour theorem)

What are the forbidden minors? K_7 is one of them.

Forbidden minors

not planar: K_5, $K_{3,3}$

not linkless:

not 4-flat:

Forbidden minors

not planar: K_5, $K_{3,3}$

not linkless: K_6

not 4-flat:

Forbidden minors

not planar: K_5, $K_{3,3}$

not linkless: K_6, $K_{3,3,1}$

not 4-flat:

Forbidden minors

not planar: K_5, $K_{3,3}$

not linkless: K_6, $K_{3,3,1}$

not 4-flat: K_7

Forbidden minors

not planar: K_5, $K_{3,3}$

not linkless: K_6, $K_{3,3,1}$

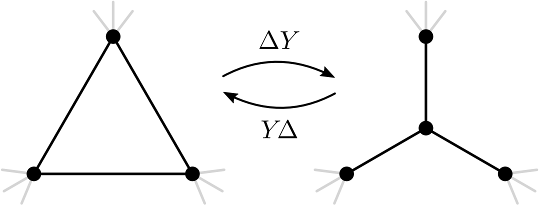
not 4-flat: K_7, $K_{3,3,1,1}$

Forbidden minors

not planar: K_5, $K_{3,3}$ + $\Delta Y/Y\Delta$-trafos

not linkless: K_6, $K_{3,3,1}$ + $\Delta Y/Y\Delta$-trafos → 7 graphs

not 4-flat: K_7, $K_{3,3,1,1}$

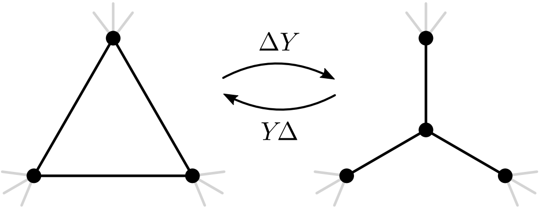


Forbidden minors

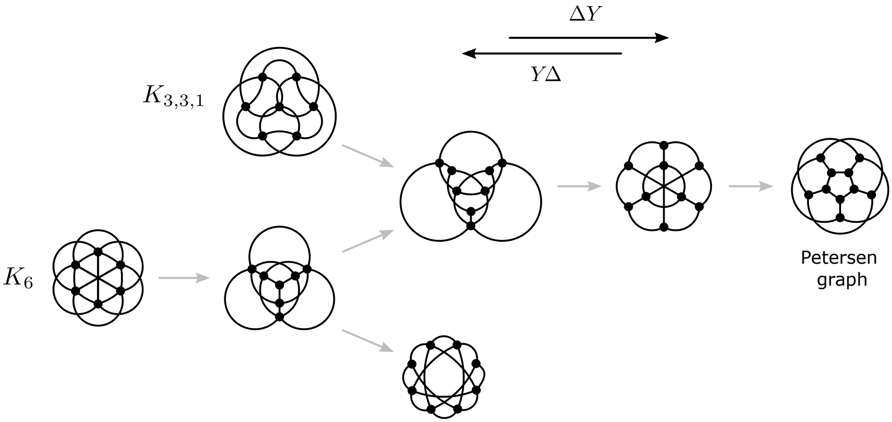
not planar: K_5, $K_{3,3}$ + $\Delta Y/Y\Delta$-trafos

not linkless: K_6, $K_{3,3,1}$ + $\Delta Y/Y\Delta$-trafos → 7 graphs

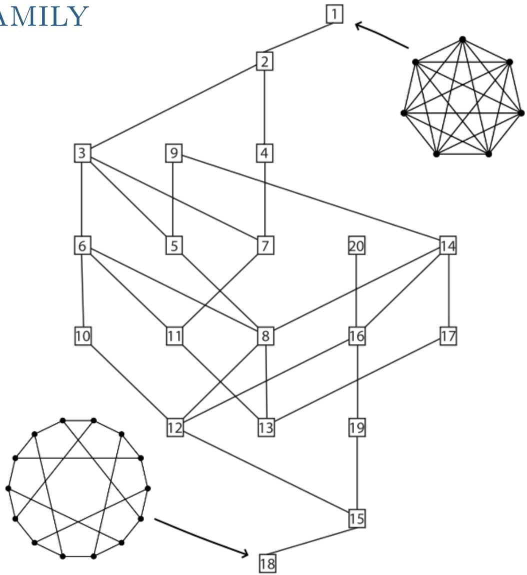
not 4-flat: K_7, $K_{3,3,1,1}$ + $\Delta Y/Y\Delta$-trafos → 76 graphs (Heawood graphs)



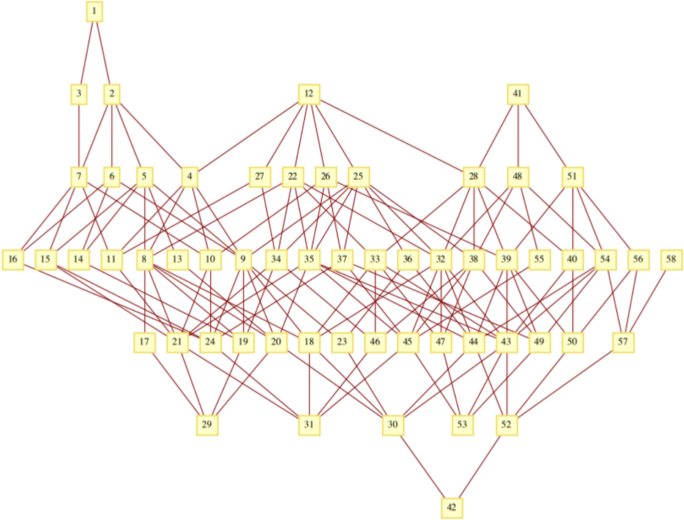
Intrinsically linked graphs



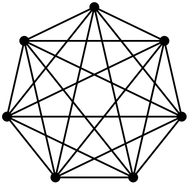
The K_7 family

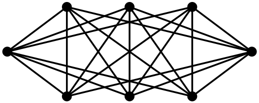


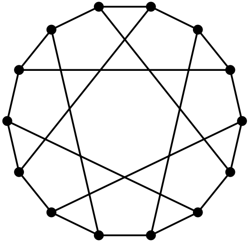
THE $K_{3,3,1,1}$ FAMILY (GOLDBERG, MATTMAN, NAIMI)



The Heawood graphs







Conjecture.

(i) *all Heawood graphs are forbidden minors.*
(ii) *there are no other forbidden minors.*

Colin de Verdière graph invariant μ

- $\mu \leq 0 \iff$ no edges
- $\mu \leq 1 \iff$ forest
- $\mu \leq 2 \iff$ outer-planar
- $\mu \leq 3 \iff$ planar
- $\mu \leq 4 \iff$ linkless
- $\mu \leq 5 \overset{?}{\iff}$ 4-flat $\quad \leftarrow$ conjectured by van der Holst

Cones over linkless graphs have $\mu \leq 5$. Heawood graphs have $\mu = 6$.

van der Holst's conjectures

Conjecture.

4-flat graphs are characterized as graphs without Heawood minors.

Conjecture.

4-flat graphs are characterized by $\mu \leq 5$.

Conjecture.

4-flat graphs are closed under ...

(i) *doubling edges*

(ii) $\Delta Y / Y\Delta$*-transformations*

$\longrightarrow$ all Heawood graphs are forbidden minors

van der Holst's conjectures

Conjecture.

4-flat graphs are characterized as graphs without Heawood minors.

Conjecture.

4-flat graphs are characterized by $\mu \le 5$.

Conjecture.

4-flat graphs are closed under ...

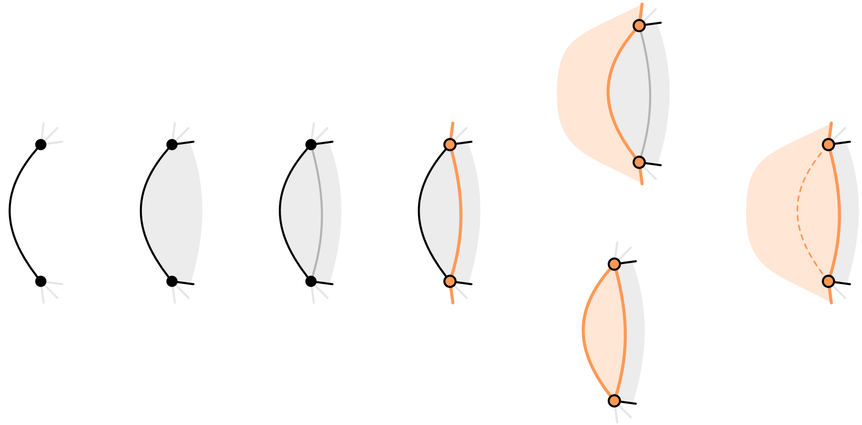
(i) *doubling edges*

(ii) $\Delta Y/Y\Delta$*-transformations*

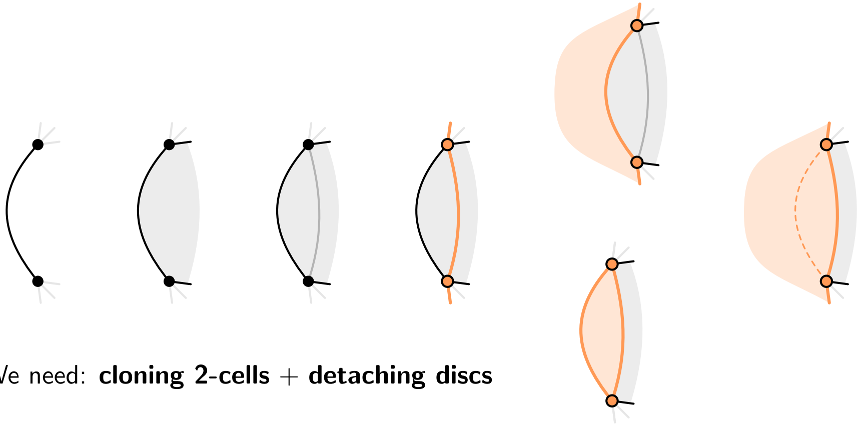
$\longrightarrow$ all Heawood graphs are forbidden minors

Operations on 2-complexes

Doubling edges preserves 4-flat



Doubling edges preserves 4-flat

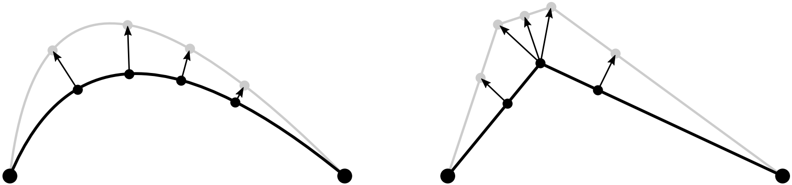


We need: **cloning 2-cells + detaching discs**

Cloning 2-cells

Given a 2-cell $c \subset C$.

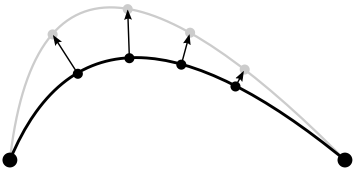
Question: can we attach (and embed) another 2-cell with boundary ∂c?

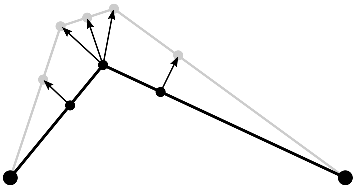


Cloning 2-cells

Given a 2-cell $c \subset C$.

Question: can we attach (and embed) another 2-cell with boundary ∂c?

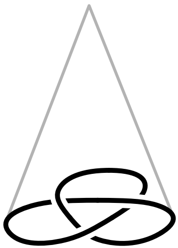




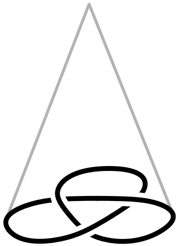
Conjecture.

Cloning is not possible as depicted here (i.e. with preserving the original 2-cell).

Try cloning this ...



Try cloning this ...



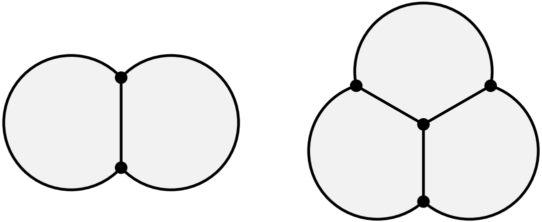
Lemma.

We can clone 2-cells, but we might need to modify the embedding of the original 2-cell.

Detaching discs

Given a disc $D \subset C$ with $\partial D \subset C^{(1)}$.

Question: Can we attach (and embed) a 2-cell with boundary ∂D?

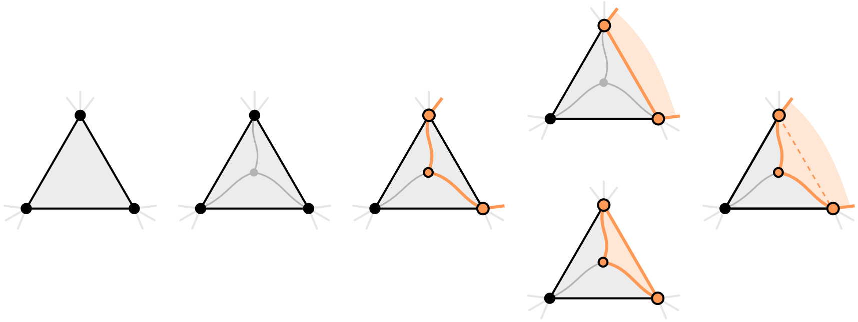


Lemma.

Yes, if D contains no vertices of C in its interior.

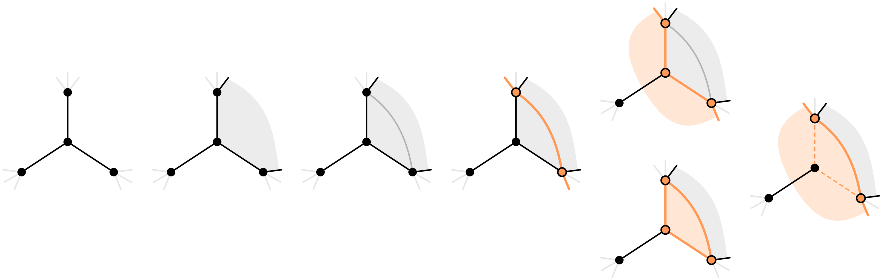
"we can detach discs from edges"

ΔY-transforms preserve 4-flat



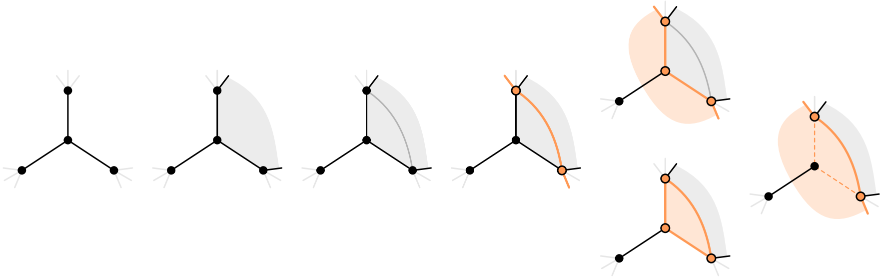
What about $Y\Delta$-transforms?

It's complicated ...



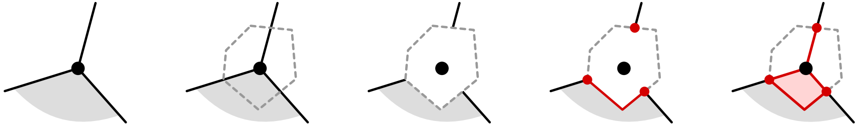
WHAT ABOUT $Y\Delta$-TRANSFORMS?

It's complicated ...

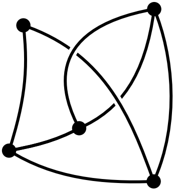


... because detaching from vertices is *complicated*.

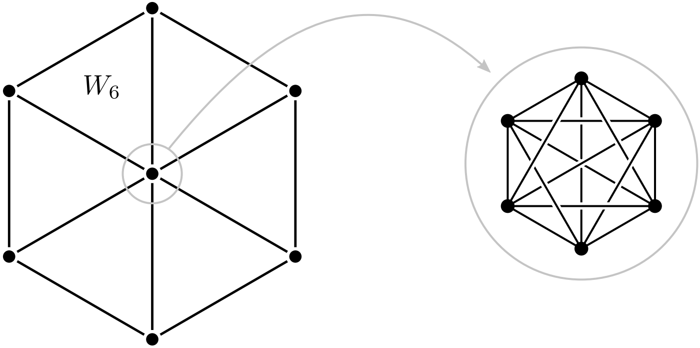
Vertex links



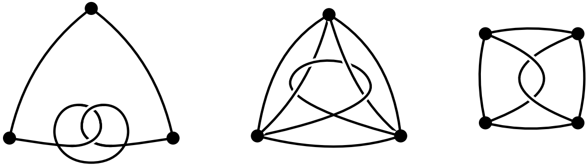
Vertex links in $\mathbb{R}^4$ live in a 3-sphere and look more like this:



Linkless links



Small links can be bad as well ...

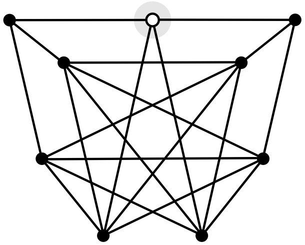


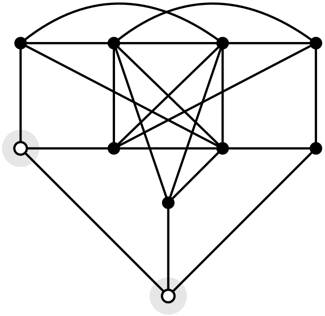
Some complexes can force bad links, even at particular 3-vertices.

Question: But can this happen in 4-flat graphs?

... whether $Y\Delta$-transforms preserve 4-flat remains **open**!

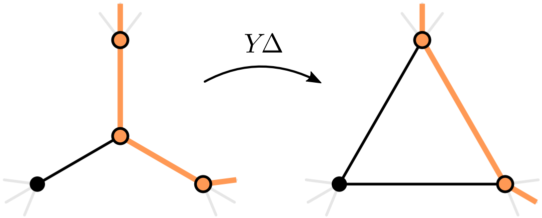
4-flat graphs with bad links





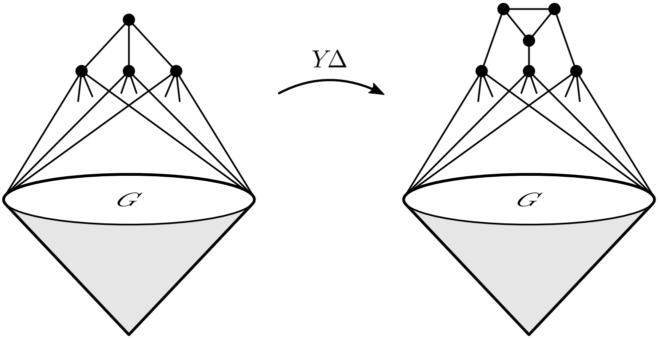
If there are 4-flat graphs with bad 3-links, then there are more forbidden minors.

$Y\Delta$-transforms for complexes



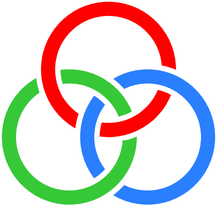
$Y\Delta$-transforms don't preserve embeddability

Let G be a graph with "unavoidable Borromean rings"

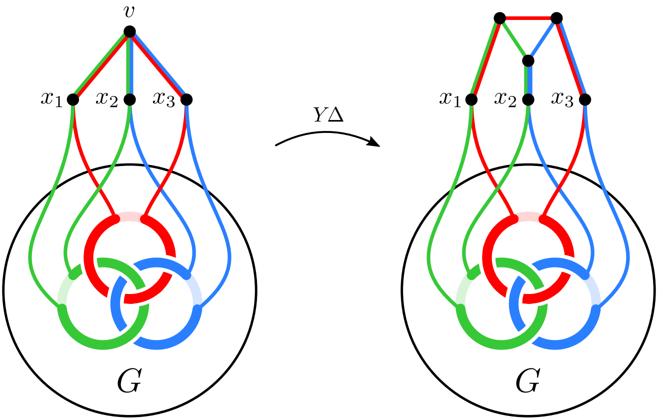


$Y\Delta$-transforms don't preserve embeddability

- Let G be a graph with "intrinsic Borromean rings"
- cone (as a graph) trice over G with apices x_1, x_2, x_3
- add a 3-vertex v adjacent to x_1, x_2, x_3
- cone (as a complex) over this graph
- for each cycle $C \subset G$, edge $e \subset C$ and pair $x_i \neq x_j$, attach a 2-cell along C but with a detour replacing $e = ab$ with the path ax_ivx_jb



$Y\Delta$-transforms don't preserve embeddability

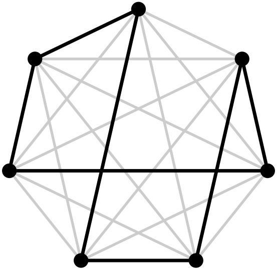


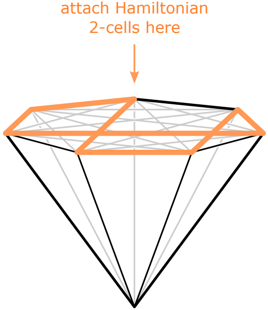
Thank you.

Bonus round: Smooth vs. PL

PL-embeddable, not smoothly embeddable

cone over K_7 + attach a 2-cell to every Hamiltonian cycle of K_7





PL-embeddable, not smoothly embeddable

- this complex embeds in $\mathbb{R}^4$
- every embedding must look essentially as follows:
  - K_7 "flat" in some 3-hyperplane
  - the cone is on one side of the hyperplane
  - the Hamiltonian 2-cells are on the other side of the hyperplane
- K_7 is intrinsically knotted: every 3-embedding contains a knotted cycle
- in fact, a knotted cycle with arf-invariant $\neq 0$
- such a knot is not smoothly slice, i.e. no smooth 2-disc with this knot at the boundary can be only on one side of $\mathbb{R}^3$.

Thank you.