Kalai's 3^d conjecture for coordinate symmetric polytopes

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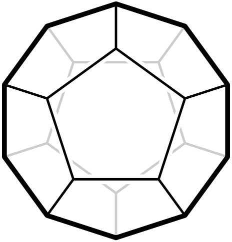
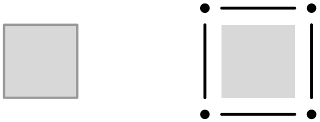
Convex polytopes

$$P = \text{conv}\{p_1, ..., p_n\} \subset \mathbb{R}^d,\ d \geq 1$$



- general dimension $d \geq 1$
- general combinatorics (not simple/simplicial etc.)
- general geometry (not lattice, etc.)

Polyhedral combinatorics



f-vector ... $\boldsymbol{f} = (f_{-1}, f_0, f_1, ..., f_{d-2}, f_{d-1}, f_d)$

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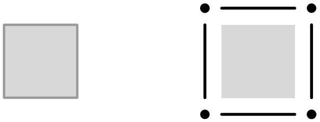

$(1, 4, 4, 1)$

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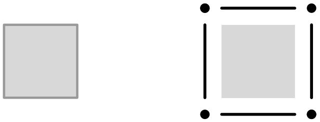

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f-vector ... $\boldsymbol{f} = (f_{-1}, f_0, f_1, ..., f_{d-2}, f_{d-1}, f_d)$

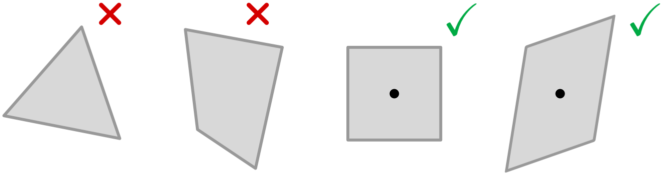
- Euler-Poincaré identity: $f_{-1} - f_0 + f_1 - \cdots + (-1)^{d+1} f_d = 0$
- Dehn-Sommerville equations: (for simplicial polytopes)

$$\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} f_j = (-1)^{d-1} f_k, \quad \text{for } k \in \{-1, ..., d-2\}$$

- lower bound theorem / upper bound theorem / g-theorem

Centrally symmetric polytopes

centrally symmetric $:\Longleftrightarrow \quad P = -P$



Kalai's 3^d conjecture

Counting faces

$$s(P) := \cancel{f_{-1}} + f_0 + f_1 + \cdots + f_{d-1} + f_d = \#\underline{\text{non}}\text{-empty faces}$$

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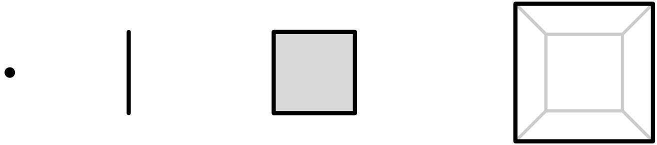
Example: d-cube $:= [-1,1]^d,\ d \geq 0$



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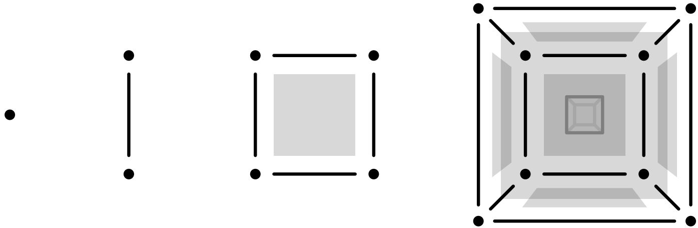

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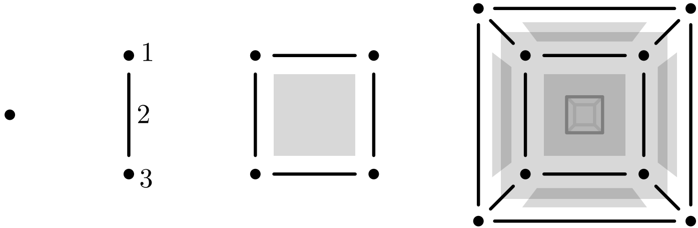

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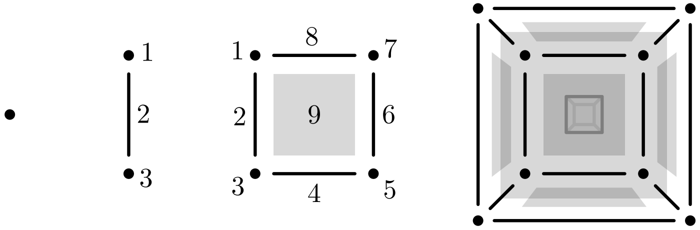

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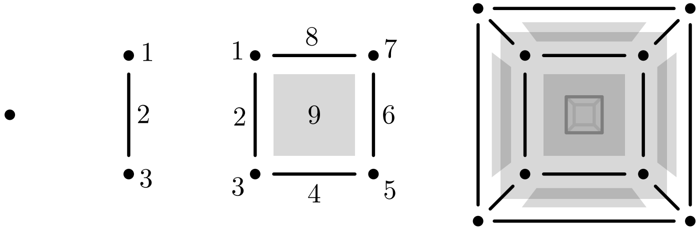

$$s(d\text{-cube}) = 3^d$$

Kalai's 3^d conjecture

$$s(P) := \cancel{f_{-1}} + f_0 + f_1 + \cdots + f_{d-1} + f_d = \#\underline{\text{non}}\text{-empty faces}$$

Conjecture. (3^d conjecture, Kalai, 1989)

For every centrally symmetric d-polytope $P \subset \mathbb{R}^d$ holds

$$s(P) \geq s(d\text{-cube}) = 3^d.$$

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But: cube is not the only minimizer! → **Hanner polytopes**

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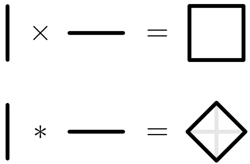
What is known ... ?

- dimension $d \leq 3$ ✓ easy
- dimension $d = 4$ ✓ not so easy (2007)
- simple/simplicial polytopes ✓ needs a lot of algebra
- without requiring central symmetry ✓ easy $\rightarrow$ $s(d\text{-simplex}) = 2^d - 1$

Hanner polytopes

Hanner polytopes are defined recursively:

(i) start from a line segment.
(ii) recursively apply Cartesian products ($\times$) and sums ($*$)



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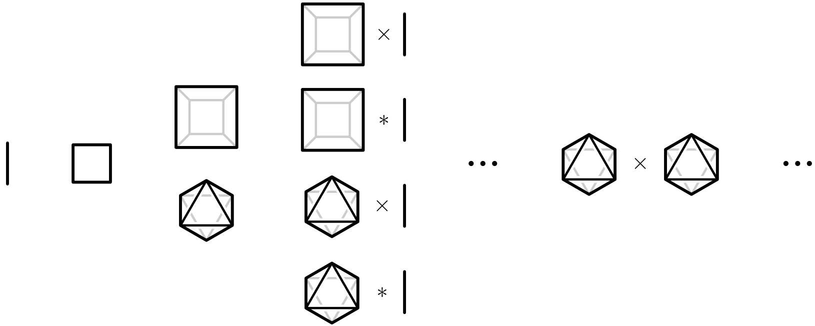
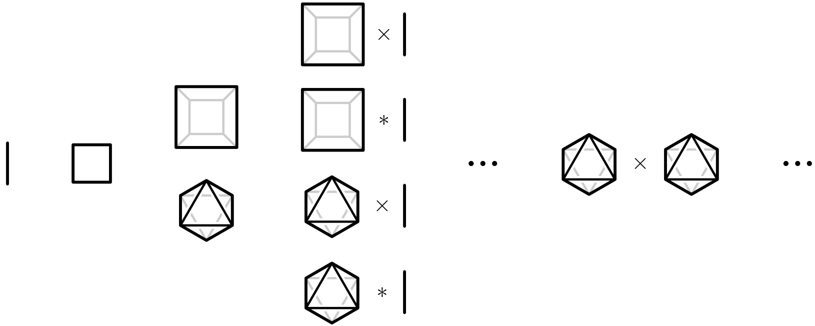


#Hanner polytopes for $d \geq 1 \ = \ 1, 1, 2, 4, 8, 18, 40, 94, 224, 548, \ldots$

Hanner polytopes



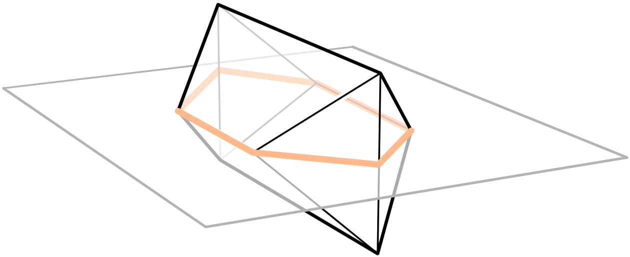
#Hanner polytopes for $d \geq 1 \;=\; 1, 1, 2, 4, 8, 18, 40, 94, 224, 548, \ldots$

Note: Hanner polytopes do not minimize face numbers per dimension.

Naive approach

Idea: induction by dimension

- slice your cs d-polytope P by a central hyperplane
  $\rightarrow$ this yields a cs $(d-1)$-polytope P'
  $\rightarrow$ P' has 3^{d-1} faces by induction hypothesis
- somehow find three times as many faces in P



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Mahler's conjecture

Mahler volume ... $M(P) := \mathrm{vol}(P) \cdot \mathrm{vol}(P^\circ)$

Conjecture. (3^d conjecture, Mahler, 1939)

For every centrally symmetric d-polytope $P \subset \mathbb{R}^d$ holds

measures "roundness" $\longrightarrow$ $M(P) \geq M(d\text{-cube}) = \dfrac{4^d}{d!}.$

But: cube is not the only minimizer! $\to$ **Hanner polytopes**

What is known ... ?

- dimension $d \leq 3$ ✓ not so easy ($d = 2$: 1939, $d = 3$: 2020)
- dimension $d = 4$? out of reach
- cube is a local minimizer ✓ (2010)
- without requiring central symmetry ? open $\to$ $M(d\text{-simplex}) = \frac{(d+1)^{d+1}}{(d!)^2}$

Polar duality and Mahler volume

Mahler volume ... $M(P) := \mathrm{vol}(P) \cdot \mathrm{vol}(P^\circ).$

---

polar dual ... $P^\circ := \{\, x \in \mathbb{R}^d \mid \langle x, y \rangle \leq 1 \text{ for all } y \in P \,\}$

Examples:

	1:1	
line segment	$\longleftrightarrow$	line segment
n-gon	$\longleftrightarrow$	n-gon (but rotated)
cube	$\longleftrightarrow$	octahedron
d-cube	$\longleftrightarrow$	d-crosspolytope
Hanner polytope	$\longleftrightarrow$	(other) Hanner polytope

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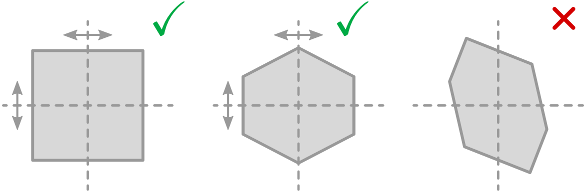
$$M(d\text{-cube}) \overset{?}{\le} M(P) \le M(d\text{-ball})$$ (Blaschke, Santaló)

(Locally) Coordinate symmetric polytopes

Coordinate symmetric polytopes

= UNCONDITIONAL POLYTOPES



coordinate symmetric $:\Longleftrightarrow$ symmetric w.r.t. all coordinate hyperplanes

Theorem. (Sanyal, W.; 2023+)

Kalai's conjecture holds for coordinate symmetric polytopes. + *minimizers*

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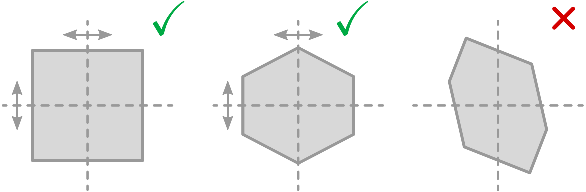

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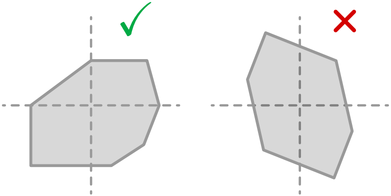
Locally coordinate symmetric polytopes

= LOCALLY ANTI-BLOCKING POLYTOPES

All discussed results hold more generally ...



locally coordinate symmetric

$:\Longleftrightarrow$ "looks" coordinate symmetric in every orthant

Locally coordinate symmetric polytopes

= locally anti-blocking polytopes

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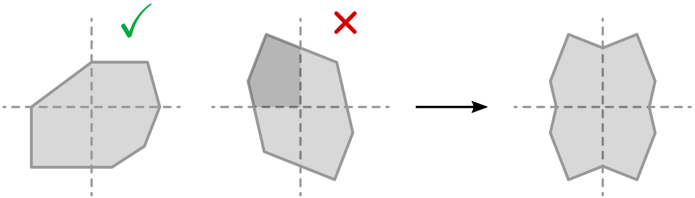

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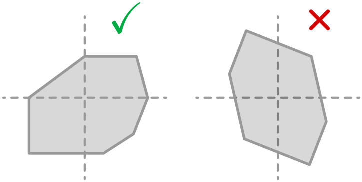
Surprise: minimizers are still coordinate symmetric for Kalai's conjecture!

But note ...



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BUT

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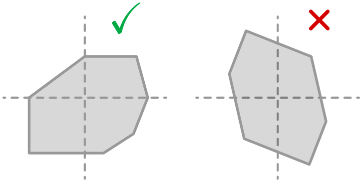

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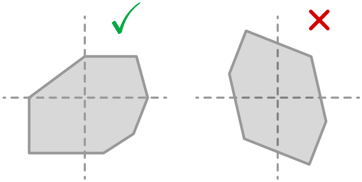

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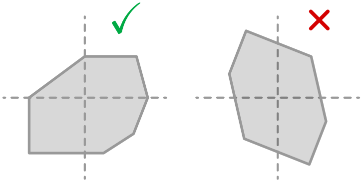

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Challenges:

- Find a polytope that is *provably* not combinatorially equivalent to a locally coordinate-symmetric polytope.
- Find many of them.
- Find many with few faces.

We have two proofs

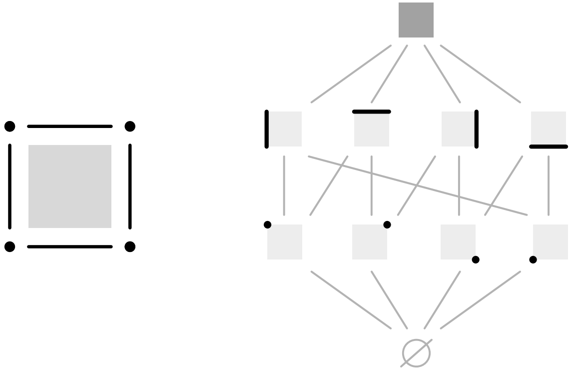
Proof I:

- mostly combinatorial
- makes the naive approach work → uses induction
- counts faces in a clever way

Proof II:

- almost entirely geometric
- reminds of the original proof for Mahler for coordinate symmetric !
- no induction !!
- we used it to classify minimizers

The face lattice



$$\mathcal{F}(P) := \{ \text{faces of } P \text{ ordered by inclusion} \}$$

The face lattice

Meet, join, intervals, ...

- $\sigma \wedge \tau :=$ largest face that contained in σ and $\tau \ = \ \sigma \cap \tau$
- $\sigma \vee \tau :=$ smallest face that contains σ and τ
- $[\sigma, \tau] :=$ faces that are contained in τ and contain σ

Properties:

- $\mathcal{F}(P^\circ)$ is "upside down" $\mathcal{F}(P)$
- face lattices are **complemented**: for $\sigma \in \mathcal{F}(P)$ exists $\tau \in \mathcal{F}(P)$ s.t.

$$\sigma \wedge \tau = \varnothing \quad \text{and} \quad \sigma \vee \tau = P.$$

- intervals in face lattices are face lattice

Kalai for coordinate symmetric polytopes (I)

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Theorem. (Sanyal, W.; 2023+)

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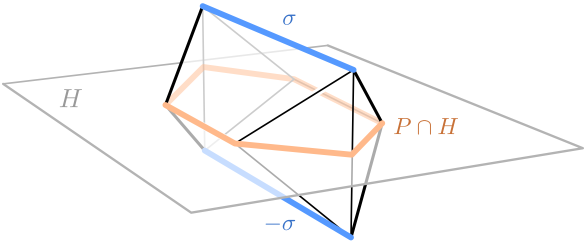

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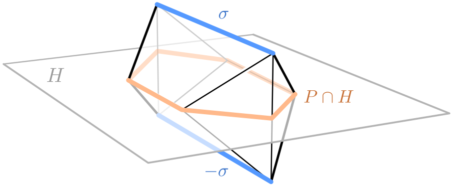

$S_+ :=$ {faces of P that intersect σ but not $-\sigma$}

$S_0 :=$ {faces of P that intersect both or neither of $\pm\sigma$}

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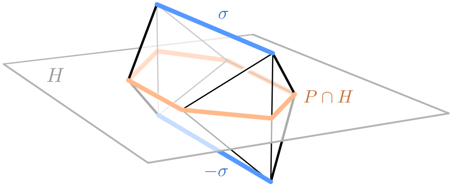

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$|S_\bullet| \geq 3^{d-1}$

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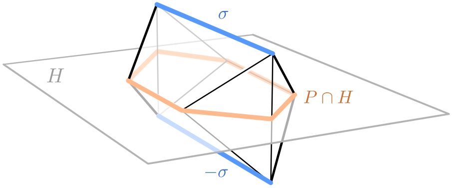

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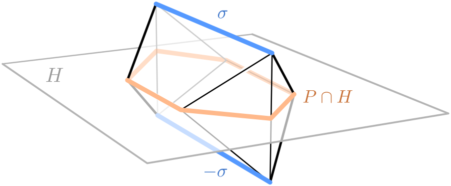

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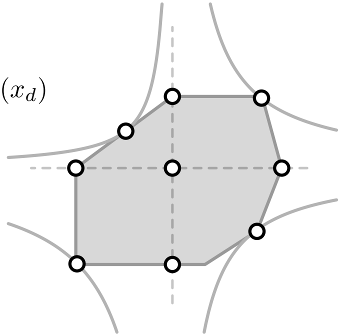
$$|S_+| \geq \Big|\Big\{\text{compl}\big(\sigma; [\check\sigma, \hat\sigma]\big) \,\Big|\, \varnothing \subset \check\sigma \subseteq \sigma \subseteq \hat\sigma \subset P\Big\}\Big|$$
$$= s\big((\varnothing, \sigma]\big) \cdot s\big([\sigma, P)\big) = s(\sigma) \cdot s(\sigma^\diamond) \overset{\text{IH}}{\geq} 3^{\dim(\sigma)} \cdot 3^{d-1-\dim(\sigma)} = 3^{d-1}$$

Kalai for coordinate symmetric polytopes (II)

$$s \in \underbrace{\{-,0,+\}^d}_{\#=3^d}, \qquad s\text{-orthant} \;\ldots\; \mathbb{R}^d_s := \{x \in \mathbb{R}^d \mid \operatorname{sign}(x) = s\}$$

1. identify one *special point* per orthant
   $=$ maximum of $s_1 \log(x_1) + \cdots + s_d \log(x_d)$ in $P \cap \mathbb{R}^d_s$
2. **show:** a face contains at most one special point



Classification of minimizers

If P has $s(P) = 3^d$ then define

$$G(P) := \begin{cases} \{1, ..., d\} \\ i \sim j \iff P \cap \{x_i, x_j = 0\} \text{ is an axis-aligned rectangle} \end{cases}$$

$G(P)$ does not contain a path of length three.

$\implies$ such graphs are *cographs*. (graph analogues of Hanner polytopes)

One can reconstruct P from $G(P)$:

$$P(G) = \mathrm{conv}\left\{ \sum_{i \in I} \pm e_i \;\middle|\; I \subseteq \{1, ..., d\} \text{ where } G[I] \text{ is a complete graph} \right\}.$$

If G is a cograph, then $P(G)$ is a Hanner polytope.

□

Kalai's flag conjecture

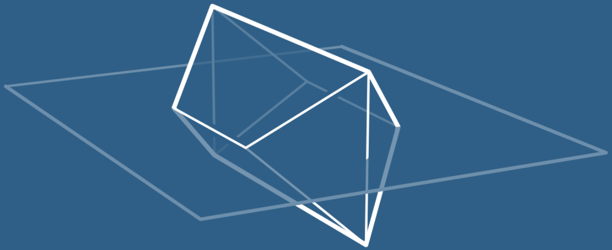
$$S(P) := \#\text{flags of } P$$

Conjecture.

For every centrally symmetric d-polytope $P \subset \mathbb{R}^d$ holds

$$S(P) \geq S(d\text{-cube}) = d!\,2^d.$$

Thank you.



R. Sanyal, M. Winter. (arXiv:2308.02909)
"Kalai's 3^d-conjecture for unconditional and locally anti-blocking polytopes".

Mahler for coordinate symmetric polytopes

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polar dual ... $P^\circ := \{\, x \in \mathbb{R}^d \mid \langle x, y \rangle \leq 1 \text{ for all } y \in P \,\}$

Theorem. (Saint-Raymond, 1980; Reisner, 1987)

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$$M(P) \geq M(d\text{-cube}) = \frac{4^d}{d!}.$$

$$\mathrm{vol}(P) = 2^d \, \mathrm{vol}(P^+)$$

.