Using Izmestiev's Theorem

– a tool in spectral polytope theory –

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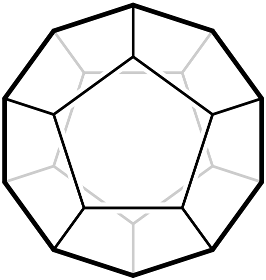
30. September, 2022

Polytope theory



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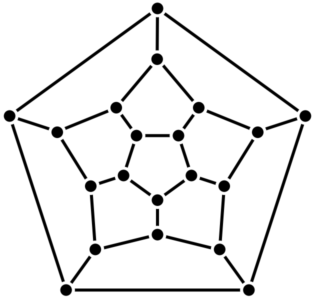
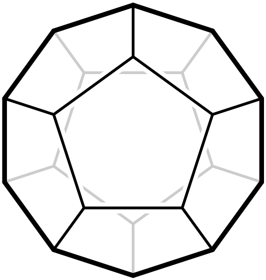


$G_P = (V, E)$

"edge-graph"

Polytope theory



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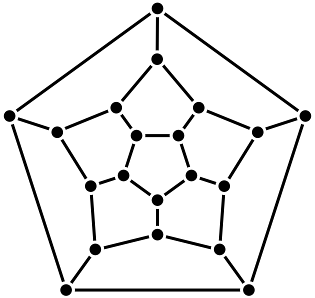
"skeleton"



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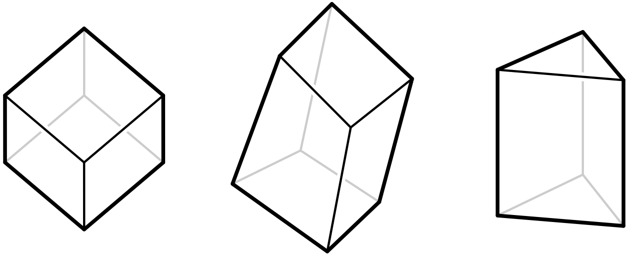
Can you reconstruct ...

- geometry?
- combinatorial type?
- dimension?



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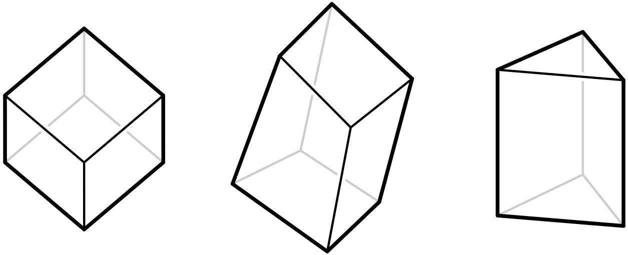

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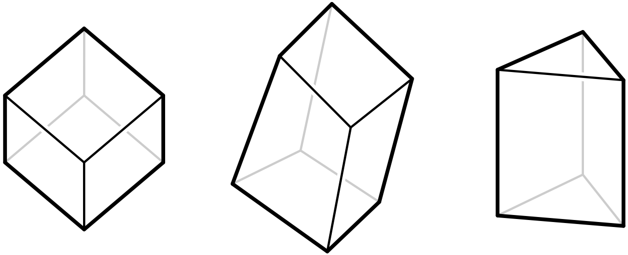

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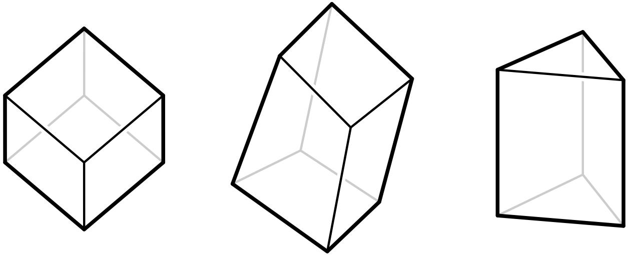

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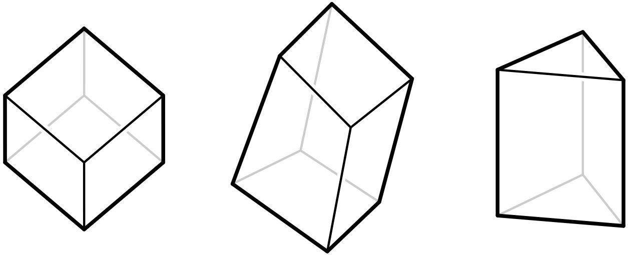

Reconstruction possible in special cases: 3-dimensional, simple, zonotopes, ...

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- simplicial + edge-graph + space of self-stesses
  $\rightarrow$ unique up to linear equivalence (Novik & Zheng, 2021)

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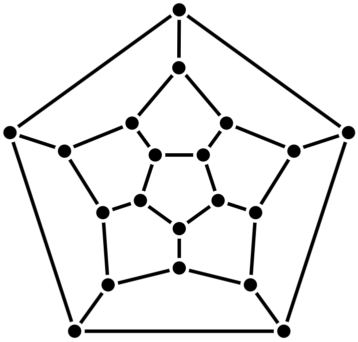
Open: does edge-graph + edge-length determine combinatorics?

Spectral graph theory



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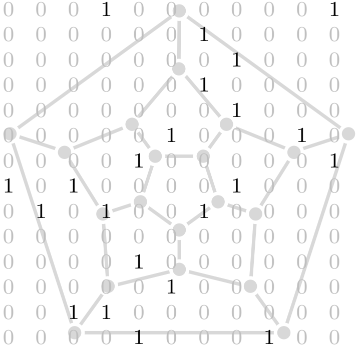

$$A_P = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&1\\
0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0\\
0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0&0\\
0&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&0&1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
1&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
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1&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1\\
0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
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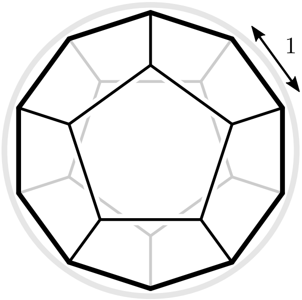
\end{bmatrix}$$

A superficial example



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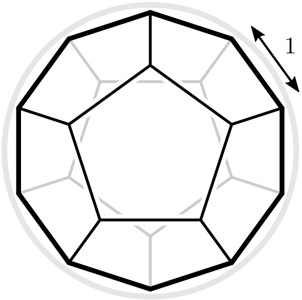

$$\mathrm{Spec}(G_P) = \{\, \theta_1 > \theta_2 > \cdots > \theta_m \,\}$$

$$r = \left(1 - \frac{\theta_2}{\deg(G)}\right)^{-1/2}$$

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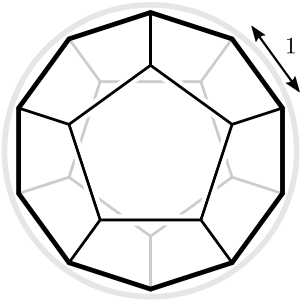

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$$r = \left(1 - \frac{\sqrt{5}}{3}\right)^{-1/2} \approx 1.4012...$$

The bias of the adjacency matrix

$$A_P = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0\\
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0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1\\
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0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0\\
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0&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
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Izmestiev's Theorem

"Polytope skeleta are spectral embeddings of the edge-graph."

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Colin de Verdière embedding

"Polytope skeleta are ~~spectral embeddings~~ of the edge-graph."

Spectral embeddings

= *"graph embeddings constructed from spectral data of generalized adjacency matrices"*

Definition.

A **generalized adjacency matrix** is a symmetric matrix $M \in \mathbb{R}^{n\times n}$ with

$$i \neq j \text{ and } ij \notin E \implies M_{ij} = 0.$$

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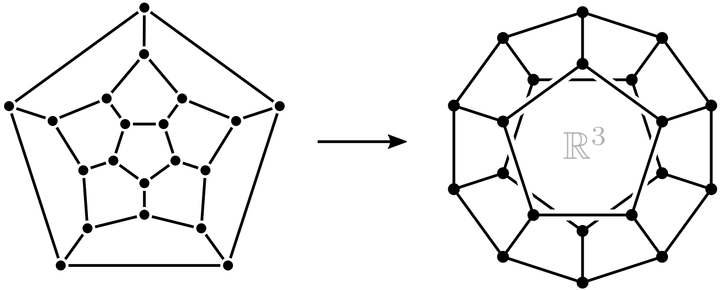

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Theorem. (Izmestiev, 2007)

If $P \subset \mathbb{R}^d$ has $0 \in \operatorname{int}(P)$, then there exists a matrix $M \in \mathbb{R}^{n\times n}$ with

(i) $M_{ij} > 0$ *whenever* $ij \in E$,

(ii) $M_{ij} = 0$ *whenever* $i \neq j$ *and* $ij \notin E$,

(iii) $\dim\ker(M) = d$,

(iv) $MX_p = 0$, *where* $X_p^\top = (p_1, ..., p_n) \in \mathbb{R}^{d\times n}$,

(v) M *has a single positive eigenvalue of multiplicity 1.*

M ... Izmestiev matrix = Alexandrov matrix of polar dual

$$M_{ij} := \frac{\partial^2 \operatorname{vol}(P^\circ(\mathbf{c}))}{\partial c_i \partial c_j}\bigg|_{\mathbf{c}=(1,...,1)} = \frac{\operatorname{vol}(e_{ij}^\circ)}{\|p_i\|\|p_j\| \sin\sphericalangle(p_i, p_j)}$$

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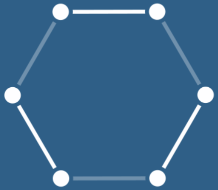
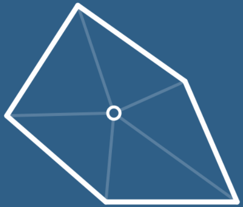
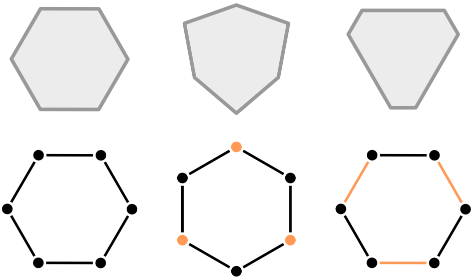




Application: capturing symmetries



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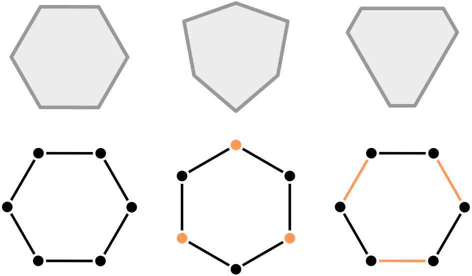

Theorem. (W., 2021)

There always exists a coloring $\mathfrak{c}\colon V \cup E \to \mathfrak{C}$ *so that* $\mathrm{Aut}(G_P^{\mathfrak{c}}) \simeq \mathrm{Aut}(P)$.

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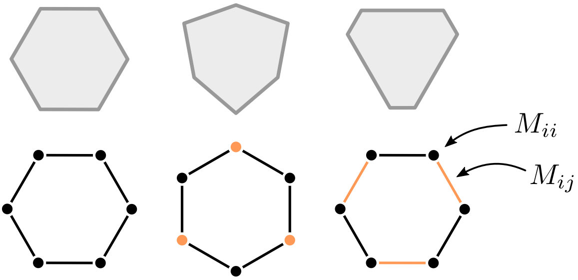

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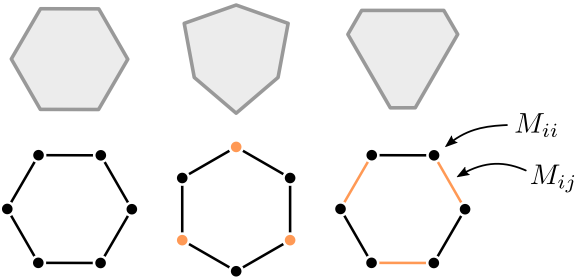

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Conjecture.

One can color edges by edge-length and vertices by distance to symmetry center.

Application: metric reconstruction

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A polytope can be uniquely reconstructed (up to isometry) from its edge-graph and its edge-length ...

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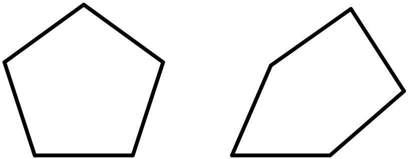
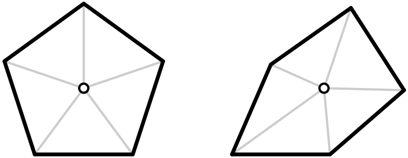


Application: metric reconstruction

Conjecture.

A polytope can be uniquely reconstructed (up to isometry) from its edge-graph, its edge-lengths, and vertex-distances to some interior point.



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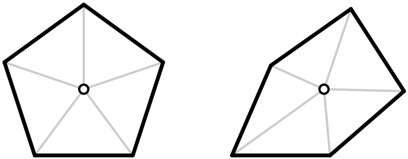
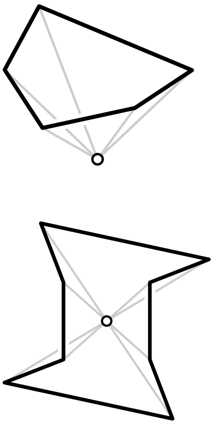


Using spectral techniques we verified the conjecture for ...

- polytopes of a fixed combinatorial type
- centrally symmetric polytopes
- small perturbations

Application: metric reconstruction



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Theorem. (W., 2022+)

Given two combinatorially equivalent polytopes $P \subset \mathbb{R}^d, Q \subset \mathbb{R}^d$ so that

- $0 \in \mathrm{int}(Q)$,
- *edges in Q are of the same length as in P, and*
- *vertex-point distances in Q are the same as in P,*

then $P \simeq Q$ (i.e. P and Q are isometric).

Application: metric reconstruction

Theorem. (W., 2022+)

Given two combinatorially equivalent polytopes $P \subset \mathbb{R}^d, Q \subset \mathbb{R}^d$ so that

- $0 \in \text{int}(Q)$,
- *edges in Q are at most as long as in P, and*
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Corollary.

The realization space of a polytope has dimension at most $f_0 + f_1 - d - 1$.

What else ...

Bounding the diameter of edge-graphs (Hirsch conjecture).

H. Narayanan, R. Shah, N. Srivastava (2022).
"A spectral approach to polytope diameter"

The Theorem of Izmestiev brings you half-way to solving ...

- a conjecture by Kalai (solved by Novik & Zheng, 2021)
- Stoker's conjecture (solved by Wang & Xie, 2022)

I. Novik, H. Zheng (2021).
"Reconstructing simplicial polytopes from their graphs and affine 2-stresses"

J. Wang, Z. Xie (2022).
"On Gromov's dihedral rigidity conjecture and Stoker's conjecture"

Thank you.

I. Izmestiev (2007).
"The Colin de Verdière number and graphs of polytopes".

M. Winter (2020).
"Eigenpolytopes, spectral polytopes and edge-transitivity".

M. Winter (2022).
"Capturing polytopal symmetries by coloring the edge-graph".

M. Winter (2023).
"Rigidity, tensegrity and reconstruction of polytopes under metric constraints".