RIGIDITY & TENSEGRITY OF FRAMEWORKS FROM CONVEX POLYTOPES

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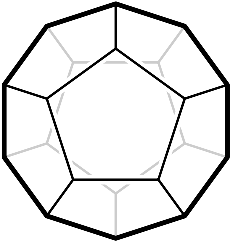
THE SETTING: CONVEX POLYTOPES

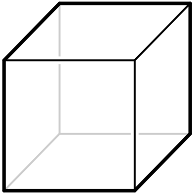
$$P = \text{conv}\{p_1, ..., p_n\} \subset \mathbb{R}^d$$



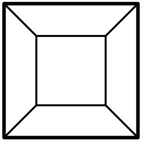
- mostly not simplicial!
- general dimension $d \geq 2$.

General theme: build a framework from the polytope → is it rigid?

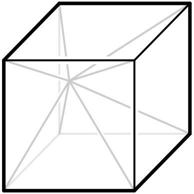
THREE TYPES OF FRAMEWORKS



flexible polytopes



Schlegel diagrams

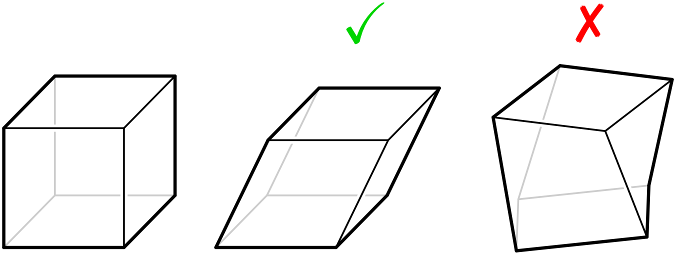


coned skeleta

FLEXIBLE POLYTOPES

with Bernd Schulze

Flexing a polytope



- preserving edge lengths

  but also

- preserve planarity of faces
- preserve convexity
- preserve combinatorial type

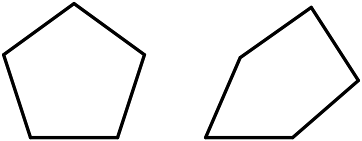
FLEXING A POLYTOPE

Examples of *rigid* polytopes:

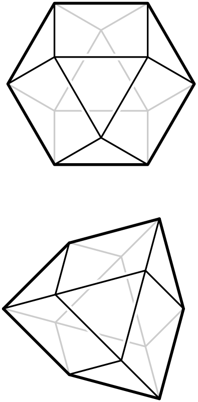
- every simplicial polytope (by Cauchy's rigidity theorem)
- every polytope with triangular 2-faces (e.g. 24-cell)

Examples of *flexible* polytopes:

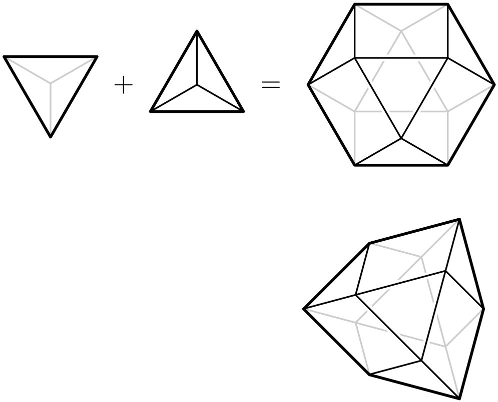
- polygons
- cubes and other prisms



Flexible polytopes



MINKOWSKI SUMS $A+B:=\{a+b \mid a\in A, b\in B\}$



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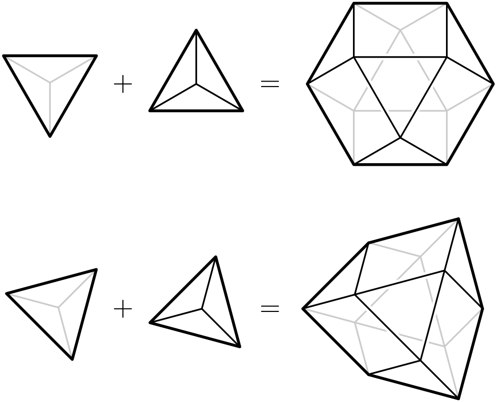

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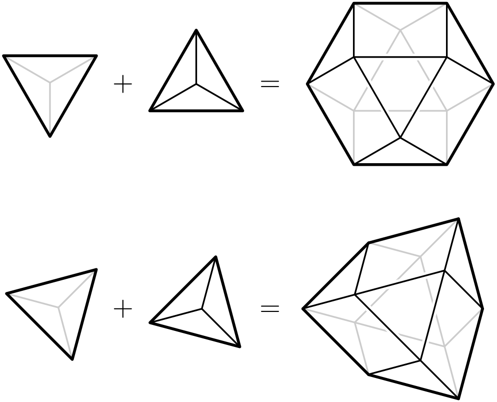
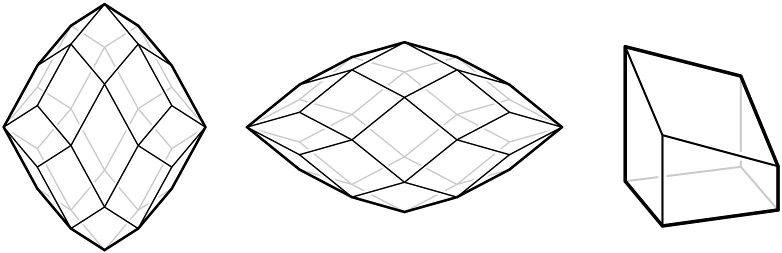


This includes all **zonotopes** $:=$ Minkowski sums of line segments

AFFINE FLEXES := A FLEX REALIZED BY AN AFFINE TRANSFORMATION



Theorem. (CONNELLY)

A framework has an affine flex $\iff$ *its edge directions lie on a conic at* ∞.

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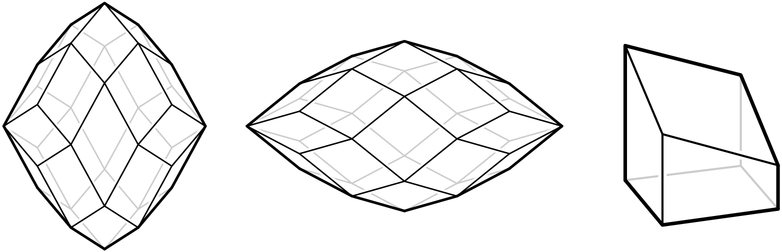

Theorem. (CONNELLY)

A framework has an affine flex $\iff$ *its edge directions lie on a conic at* ∞.

For example, if there are at most 5 edge directions (in 3D).

SUMMARIZING …

We know the following classes of flexible polytopes

- polygons
- Minkowski sums
- all edges on a conic at ∞ (e.g. at most five edge directions)

Question: Are there any others?

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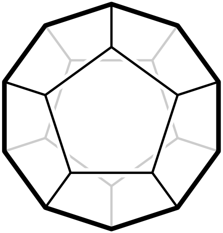
Some other facts:

- $d = 3$: DOF minus constraints $= 0$
- can be transformed into a pure framework (no coplanarity constraints)
- for all known cases, flexibility is preserved under affine transformations

TOY EXAMPLE: THE REGULAR DODECAHEDRON



Question: Is the regular dodecahedron rigid?

- *very* likely yes, but a rigorous argument is missing
- 5-dimensional space of non-trivial infinitesimal flexes
- infinitesimal flexes vanish for any linear transformation (that I tried)

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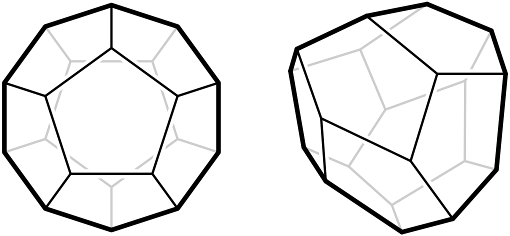

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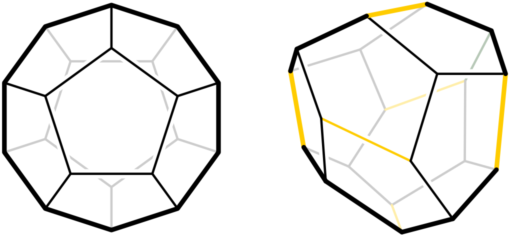

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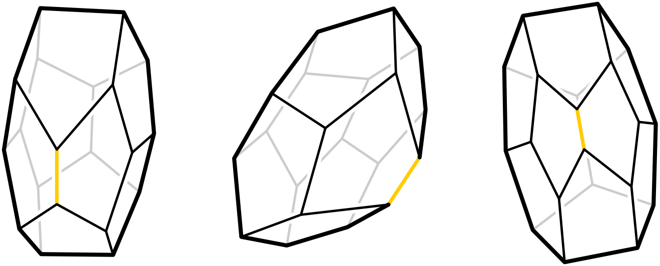
A PHYSICAL MODEL

SPECIAL CASE: 3D

The realization space of a 3-polytope is a smooth variety of dimension $E + 6$

Lemma.

P is rigid $\iff$ every perturbation of edge lengths is realizable



"CONJECTURES"

Conjecture.

P flexible $\Longrightarrow$ all linear transformations of P are flexible.

Conjecture.

P flexible $\Longleftrightarrow$ all linear transformations of P are infinitesimally flexible.

Conjecture.

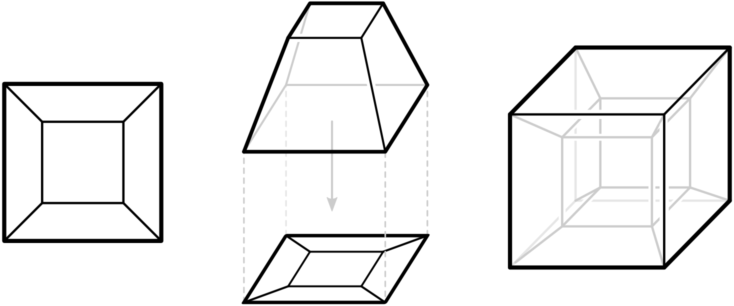
A simple polytope is generically rigid.

simple in $\mathbb{R}^d$ = regular of vertex degree d

SCHLEGEL DIAGRAMS & OTHER PROJECTIONS

with Bernd Schulze

SCHLEGEL DIAGRAM := SPECIAL PROJECTION OF A POLYTOPE



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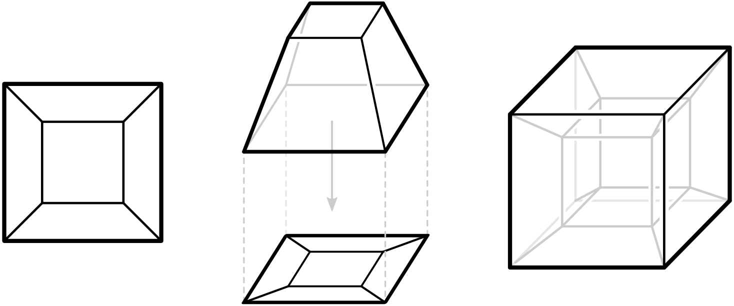

Question: is it always (locally) rigid?

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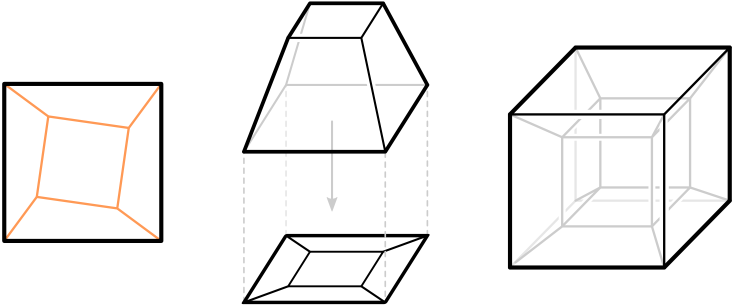

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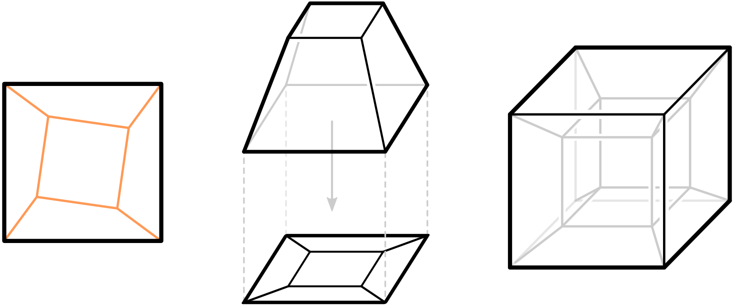
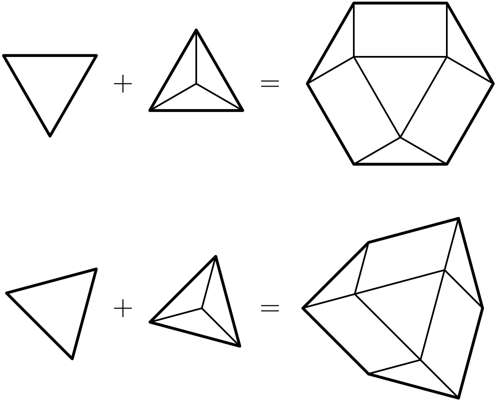


Question: is it always (locally) rigid? **No!**

FLEXIBLE SCHLEGEL DIAGRAMS



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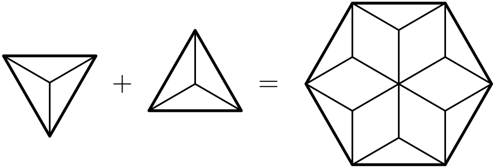

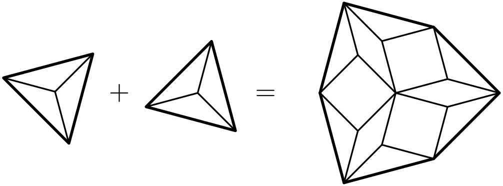

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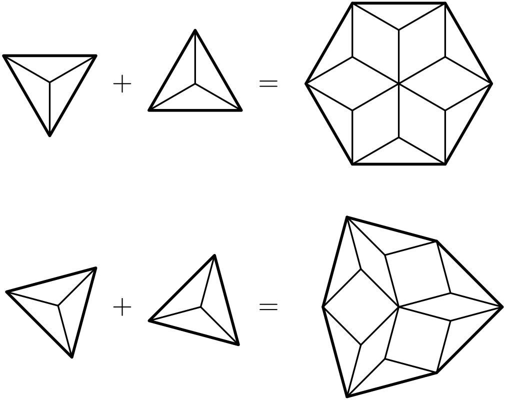
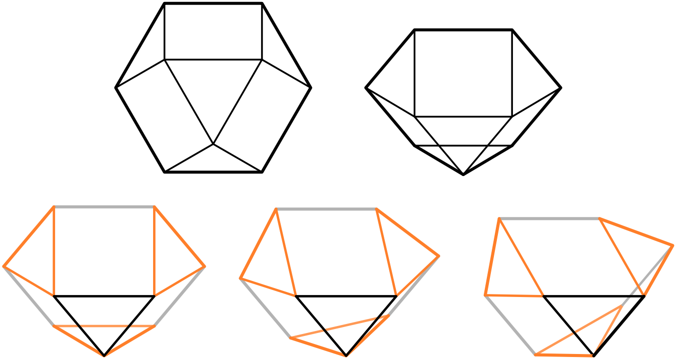


Question: Is flexibility independent of the projection direction?

"CONJECTURES"

Conjecture.

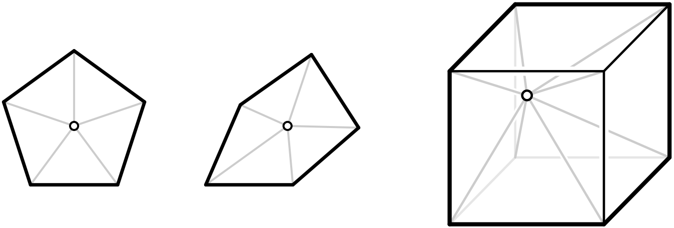
Flexibility of projections is (almost) independent of the projection direction.



CONED SKELETA

"Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints"
(`arXiv:2302.14194`)

POINTED POLYTOPES AND CONED SKELETA



- edges as bars + central bars (special case of Schlegel diagram)

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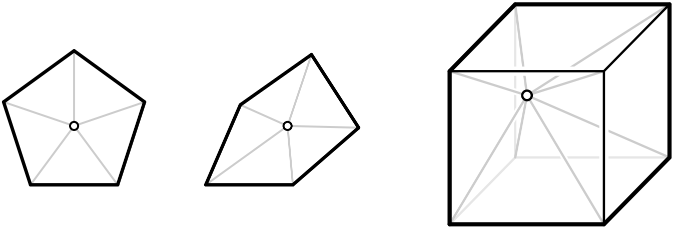

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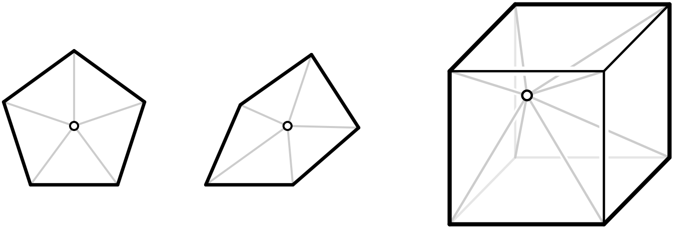

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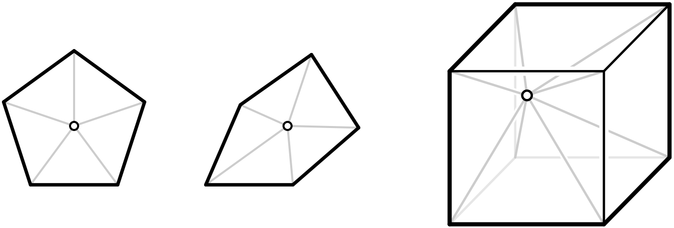

- edges as bars + central bars (special case of Schlegel diagram)
- **tensegrity:** edges as cables + central struts

Question: is it always (locally) rigid? (not always infinitesimally rigid)

RIGIDITY OF CONED SKELETA

Theorem. (W., 2023)

If the point is chosen from the interior of the polytope, then

(i) *as a framework, it is locally rigid* *(actually, prestress stable)*

(ii) *as a polytope, it is globally rigid*

(iii) *as a centrally symmetric framework, it is universally rigid*

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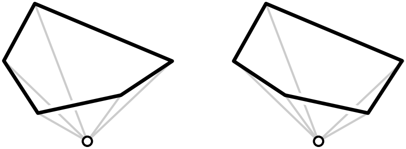

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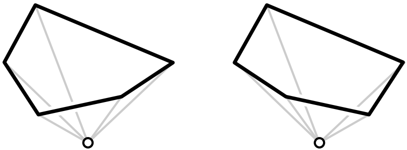

THE MAIN CONJECTURE

Conjecture. (W., 2023)

If $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ are pointed polytopes with the same edge-graph, s.t.

(i) $x_Q \in Q$ *is an <u>interior</u> point,*

(ii) *edges in Q are at most as long as in P,*

(iii) *vertex-point distances in Q are at least as large as in P,*

then P and Q are isometric.

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then P *and* Q *are isometric.*

We proved this in three special cases:

- P and Q are "sufficiently close"
- P and Q are combinatorially equivalent
- P and Q are centrally symmetric

INGREDIENTS TO THE PROOF

convex geometry $+$ spectral graph theory

I. Izmestiev (2007), *“The Colin de Verdière number and graphs of polytopes”*

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$$P^\circ(\mathbf{c}) := \big\{x \in \mathbb{R}^d \mid \langle x, p_i\rangle \le c_i \text{ for all } i \in V(G_P)\big\}.$$

Expand $\mathrm{vol}(P^\circ(\mathbf{c}))$ at $\mathbf{c} = \mathbf{1}$:

$$\mathrm{vol}(P^\circ(\mathbf{c})) = \mathrm{vol}(P^\circ) + \langle \underset{\substack{\uparrow \\ \text{Wachspress} \\ \text{coordinates}}}{\tilde\alpha}, \mathbf{c} - \mathbf{1}\rangle + \tfrac{1}{2}(\mathbf{c} - \mathbf{1})^\top \underset{\substack{\uparrow \\ \text{Izmestiev} \\ \text{matrix}}}{\tilde M}(\mathbf{c} - \mathbf{1}) + \cdots$$

These objects define a stress matrix that certifies prestress stability.

More conjectures

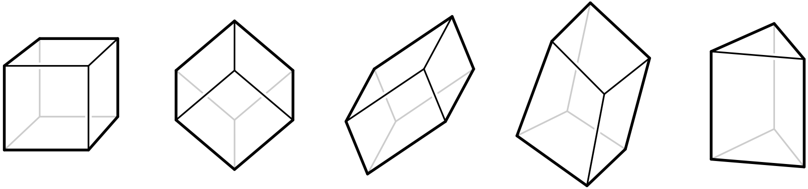
MORE CONJECTURES

Conjecture.

The edge-graph and edge lengths determine the combinatorial type.

Conjecture. (strengthening Cauchy's rigidity theorem)

A polytope is uniquely determined by its 2-skeleton and the shape of its 2-faces.



Thank you.