Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints

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The setting: convex polytopes

\[ P = \text{conv}\{p_1, \ldots, p_n\} \subset \mathbb{R}^d \]

- always convex!
- general dimension \( d \geq 2 \)
- general geometry & combinatorics (not always simple/simplicial/lattice/...)
- always of full dimension
Combinatorics of polytopes

$G_P := \{ \text{vertices and edges of } P \}$

$\mathcal{F}(P) := \{ \text{faces of } P \text{ ordered by inclusion} \}$
“In how far is a polytope determined by partial combinatorial and geometric data, up to isometry, affine transformation or combinatorial equivalence?”
Reconstruction of polytopes

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\[ \mathcal{F}(P) \simeq \mathcal{F}(Q) \]
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partial data \quad \longrightarrow \quad \text{combinatorics} \quad \longrightarrow \quad \text{geometry}
Example: edge-graph & edge lengths

Simple polytopes:
- combinatorics can be reconstructed (Blind & Mani; Kalai)
- geometry cannot be reconstructed

Simplicial polytopes:
- geometry can be reconstructed, once combinatorics in known (Cauchy)
- combinatorics cannot always be reconstructed (cyclic polytopes)

... what additional data is needed to permit a reconstruction?
Two topics

flexible polytopes

pointed polytopes
FLEXIBLE POLYTOPES
(with Bernd Schulze)
Flexible polytopes

Flexing a polytope

- preserving edge lengths
- but also
- preserve planarity of faces
- preserve convexity
- preserve combinatorial type
Flexible polytopes

Flexing a polytope

Examples of rigid polytopes:
- every simplicial polytope (by Cauchy’s rigidity theorem)
- every polytope with triangular 2-faces (e.g. 24-cell)

Examples of flexible polytopes:
- polygons
- cubes and other prisms
Flexible polytopes

Minkowski sums

\[ A + B := \{ a + b | a \in A, b \in B \} \]

This includes all zonotopes := Minkowski sums of line segments
Minkowski sums  

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Minkowski sums \[ A + B := \{ a + b \mid a \in A, b \in B \} \]

This includes all zonotopes \( := \) Minkowski sums of line segments
**Affine flexes** := flexes realized by affine transformations

**Theorem.** (Connelly, Gortler, Theran, 2018)

A framework has an affine flex \(\iff\) its edge directions lie on a conic at \(\infty\).
**AFFINE FLEXES** \( := \) FLEXES REALIZED BY AFFINE TRANSFORMATIONS

**Theorem.** (Connelly, Gortler, Theran, 2018)

*A framework has an affine flex \( \iff \) its edge directions lie on a conic at \( \infty \).*

**For example**, if there are at most 5 edge directions (in 3D).
We know the following classes of flexible polytopes

- polygons
- Minkowski sums
- all edges on a conic at $\infty$ (e.g. at most five edge directions)

**Question:** Are there any others?
Summarizing ...

We know the following classes of flexible polytopes

▶ polygons
▶ Minkowski sums
▶ all edges on a conic at $\infty$ (e.g. at most five edge directions)

Question: Are there any others?

Some other facts:

▶ $d = 3$: DOF minus constraints $= 0$
▶ for all known cases, flexibility is preserved under affine transformations
**Toy example: the regular dodecahedron**

**Question:** Is the regular dodecahedron rigid?

- probably, but *we don’t know*
- 5-dimensional space of non-trivial infinitesimal flexes
- infinitesimal flexes vanish for any linear transformation (that I tried)
Toy example: the regular dodecahedron

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**Toy example: the regular dodecahedron**

**Question:** Is the regular dodecahedron rigid?

- probably, but *we don’t know*
- 5-dimensional space of non-trivial infinitesimal flexes
- infinitesimal flexes vanish for any linear transformation (that I tried)
Is the dodecahedron flexible (as a polytope with fixed edge-lengths)?

Consider the \textit{(regular) dodecahedron} \( D \subset \mathbb{R}^3 \). I want to continuously deform it so that throughout the deformation

1. it stays a convex polytope,
2. it stays a combinatorial dodecahedron (i.e. its edge-graph does not change), and
3. all edge lengths stay the same.

\textit{Can I do this?} If No, can I do it for some other realizations of the dodecahedron that is not necessarily regular? If Yes, is this possible for \textit{all} realizations?

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{dodecahedron.png}
\caption{Dodecahedron}
\end{figure}
A physical model ...
RIGIDITY OF
POINTED POLYTOPES

“Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints”
(arXiv:2302.14194)
Pointed polytopes

\[ := \text{polytope } P \subset \mathbb{R}^d + \text{point } x_P \in \mathbb{R}^d \]

**Conjecture. (W., 2023)**

If \( x_P \in \text{int}(P) \), then a pointed polytope is uniquely determined (up to isometry) by its edge-graph, edge lengths and vertex-point distances.

implies e.g. reconstruction of matroids from base exchange graph
Point in the interior is necessary ... 

Conjecture. (W., 2023)

If $x_P \in \text{int}(P)$, then a pointed polytope is uniquely determined (up to isometry) by its edge-graph, edge lengths and vertex-point distances.
TENSEGRITY VERSION

Conjecture. (W., 2023)

If \( P \subset \mathbb{R}^d \) and \( Q \subset \mathbb{R}^e \) are pointed polytopes with the same edge-graph and

(i) \( x_Q \in \text{int}(Q) \)
(ii) edges in \( Q \) are at most as long as in \( P \),
(iii) vertex-point distances in \( Q \) are at least as large as in \( P \),

then \( P \) and \( Q \) are isometric.

“\textit{A polytope cannot become larger if all its edges become shorter.}”
Main result (W., 2023)

The conjecture holds in the following cases:

I. $Q$ is a small perturbation of $P$
   - one can replace $Q$ by a graph embedding $q: G_P \to \mathbb{R}^d$
   - locally rigid as a framework (actually prestress stable)

II. $P$ and $Q$ are combinatorially equivalent
   - globally rigid as a polytope

III. $P$ and $Q$ are centrally symmetric
   - one can replace $Q$ by a centrally symmetric graph embedding $q: G_P \to \mathbb{R}^e$
   - universally rigid as a framework
NOT GLOBALLY RIGID AS A FRAMEWORK
**Warmup: simplices**

$P, Q \subset \mathbb{R}^d$ simplices,

(i) $0 \in \text{int}(Q)$,

(ii) edges in $Q$ are at most as long as in $P$.

(iii) vertex-origin distances $Q$ are at least as large as in $P$. 

Therefore $P \simeq Q$. 

□
Warmup: simplices

$P, Q \subset \mathbb{R}^d$ simplices,

(i) $0 \in \text{int}(Q)$,

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(iii) vertex-origin distances $Q$ are at least as large as in $P$.

Proof.
Warmup: simplices

\[ P, Q \subset \mathbb{R}^d \text{ simplices,} \]

(i) \( 0 \in \text{int}(Q) \), \( \implies \) \( 0 = \sum \alpha_i q_i \) ... convex combination

(ii) edges in \( Q \) are at most as long as in \( P \).

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Proof.

$$\sum \alpha_i \|p_i\|^2 = \left\| \sum \alpha_i p_i \right\|^2 + \frac{1}{2} \sum \alpha_i \alpha_j \|p_i - p_j\|^2$$
**Warmup: simplices**

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**Proof.**

\[
\sum_i \alpha_i \|p_i\|^2 = \left\| \sum_i \alpha_i p_i \right\|^2 + \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|p_i - p_j\|^2
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Therefore $P \simeq Q$. □
Warmup: simplices

Let $P, Q \subset \mathbb{R}^d$ simplices,

(i) $0 \in \text{int}(Q), \quad \implies 0 = \sum_i \alpha_i q_i \ldots$ convex combination

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\sum_i \alpha_i \|p_i\|^2 = \left\| \sum_i \alpha_i p_i \right\|^2 + \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|p_i - p_j\|^2 \\
\sum_i \alpha_i \|q_i\|^2 = \left\| \sum_i \alpha_i q_i \right\|^2 + \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|q_i - q_j\|^2
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Therefore $P \simeq Q$. \[\square\]
Expansion of polytopes

Fix $\alpha \in \Delta_n := \{(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n_{\geq 0} \mid \alpha_1 + \cdots + \alpha_n = 1\}$

$\alpha$-expansion: $\|P\|_\alpha^2 := \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|p_i - p_j\|^2$
**Expansion of Polytopes**

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\| P \|_\alpha^2 := \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \| p_i - p_j \|^2
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"If edges shrink, then the expansion decreases."
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“If edges shrink, then the expansion decreases, if $\alpha$ is chosen suitably.”
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\[ \text{\texttt{\textalpha-expansion:}} \quad \|P\|_\alpha^2 := \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|p_i - p_j\|^2 \]

“If edges shrink, then the expansion decreases, if $\alpha$ is chosen suitably.”

**Theorem. (W., 2023)**

Let $P \subset \mathbb{R}^d$ be a polytope and $q : G_P \rightarrow \mathbb{R}^e$ an embedding of its edge-graph with edges at most as long as in $P$. If $\alpha \in \Delta_n$ are Wachspress coordinates of some interior point of $P$, then

\[ \|P\|_\alpha \geq \|q\|_\alpha. \]

Equality holds if and only if $q$ is an affine transformation of the skeleton of $P$ and has the same edge lengths.
Rigidity of pointed polytopes

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Ingredients

convex geometry + spectral graph theory

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convex geometry + spectral graph theory


\[ P^\circ(c) := \{ x \in \mathbb{R}^d | \langle x, p_i \rangle \leq c_i \text{ for all } i \in V(G_P) \}. \]
**Ingredients**

**convex geometry + spectral graph theory**


\[ P^\circ(c) := \{x \in \mathbb{R}^d | \langle x, p_i \rangle \leq c_i \text{ for all } i \in V(G_P) \}. \]

Expand \( \text{vol}(P^\circ(c)) \) at \( c = 1 \):

\[ \text{vol}(P^\circ(c)) = \text{vol}(P^\circ) + \langle \tilde{\alpha}, c - 1 \rangle + \frac{1}{2}(c - 1)^\top \tilde{M}(c - 1) + \cdots \]

\( \uparrow \) Wachspress coordinates

\( \uparrow \) Izmestiev matrix
Recalling the statement

Theorem. (W., 2023)

Let $P \subset \mathbb{R}^d$ be a polytope and $q : G_P \to \mathbb{R}^e$ an embedding of its edge-graph with edges at most as long as in $P$. If $\alpha \in \Delta_n$ are Wachspress coordinates of some interior point of $P$, then

$$\|P\|_{\alpha} \geq \|q\|_{\alpha}.$$

Equality holds if and only if $q$ is an affine transformation of the skeleton of $P$ and has the same edge lengths.

$$\max_{q_1, \ldots, q_n \in \mathbb{R}^n} \|q\|_{\alpha}$$

s.t. $\|q_i - q_j\| \leq \|p_i - p_j\|$, for all $ij \in E$
Proof: via semidefinite programming

\[
\begin{align*}
\max & \quad \|q\|_{\alpha} \\
\text{s.t.} & \quad \|q_i - q_j\| \leq \|p_i - p_j\|, \quad \text{for all } ij \in E \\
& \quad q_1, \ldots, q_n \in \mathbb{R}^n
\end{align*}
\]
**Proof: via semidefinite programming**

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\end{align*}
\]

\[\downarrow\]

\[
\begin{align*}
\text{max} & \quad \sum_i \alpha_i \|q_i\|^2 \\
\text{s.t.} & \quad \sum_i \alpha_i q_i = 0 \\
& \quad \|q_i - q_j\| \leq \|p_i - p_j\|, \quad \text{for all } ij \in E \\
& \quad q_1, \ldots, q_n \in \mathbb{R}^n
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\end{align*}
\]

\[\downarrow\]

\[
\begin{align*}
\text{min} & \quad \sum_{ij \in E} w_{ij} \|p_i - p_j\|^2 \\
\text{s.t.} & \quad L_w - \text{diag}(\alpha) + \mu \alpha \alpha^\top \succeq 0 \\
& \quad w \geq 0, \mu \text{ free}
\end{align*}
\]
Proof: via semidefinite programming

\[ \|P\|_\alpha = \max_{q_1, \ldots, q_n \in \mathbb{R}^n} \|P\|_\alpha \]
\[ \text{s.t. } \|q_i - q_j\| \leq \|p_i - p_j\|, \quad \text{for all } ij \in E \]

\[ \Downarrow \]

\[ \max \sum_i \alpha_i \|q_i\|^2 \]
\[ \text{s.t. } \sum_i \alpha_i q_i = 0 \]
\[ \|q_i - q_j\| \leq \|p_i - p_j\|, \quad \text{for all } ij \in E \]
\[ q_1, \ldots, q_n \in \mathbb{R}^n \]

\[ \Downarrow \]

\[ \|P\|_\alpha = \min \sum_{ij \in E} M_{ij} \|p_i - p_j\|^2 \]
\[ \text{s.t. } L_w - \text{diag}(\alpha) + \mu \alpha \alpha^\top \succeq 0 \]
\[ w \geq 0, \mu \text{ free} \]
Consequences

Corollary.

A polytope is uniquely determined (up to affine transformations) by its edge-graph, edge lengths and the Wachspress coordinates of any interior point.

\[ \alpha_i = \frac{\text{vol}(F_i)}{\|p_i\| \text{ vol}(P)} \]
Rigidity of pointed polytopes

Are we done ... ?

\[ \sum_i \alpha_i \| p_i \|^2 = \left\| \sum_i \alpha_i p_i \right\|^2 + \| P \|_\alpha \]

\[ \forall i \quad \forall i \quad \forall i \]

\[ \sum_i \alpha_i \| q_i \|^2 = \left\| \sum_i \alpha_i q_i \right\|^2 + \| Q \|_\alpha \]

Theorem. (W., 2023)

Let \( P \subset \mathbb{R}^d \) be a polytope and \( q: G_P \to \mathbb{R}^e \) an embedding of its edge-graph with edges at most as long as in \( P \). If \( \alpha \in \Delta_n \) are Wachspress coordinates of some interior point of \( P \), then

\[ \| P \|_\alpha \geq \| q \|_\alpha. \]
Are we done ... ?

\[ \sum_i \alpha_i \|p_i\|^2 = \left\| \sum_i \alpha_i p_i \right\|^2 + \|P\|_\alpha \]

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\[ \sum_i \alpha_i \|q_i\|^2 = \left\| \sum_i \alpha_i q_i \right\|^2 + \|Q\|_\alpha \]

**Theorem. (W., 2023)**

Let \( P \subset \mathbb{R}^d \) be a polytope and \( q : G_P \rightarrow \mathbb{R}^e \) an embedding of its edge-graph with edges at most as long as in \( P \). If \( \alpha \in \Delta_n \) are *Wachspress coordinates* of some interior point of \( P \), then

\[ \|P\|_\alpha \geq \|q\|_\alpha. \]
**The Wachspress map** \( \phi: P \rightarrow Q \)

\[
\phi(x) := \sum_i \alpha_i(x)q_i.
\]

**Question:** When do we have \( \phi(x) = 0 \) for some \( x \in \text{int}(P) \)?
The Wachspress map $\phi: P \to Q$

$$\phi(x) := \sum_i \alpha_i(x)q_i.$$ 

**Question:** When do we have $\phi(x) = 0$ for some $x \in \text{int}(P)$?

Centrally symmetric

$\rightarrow \phi(0) = 0.$

Small perturbations

$\rightarrow$ if $0 \in B_\epsilon(0) \subset P$, then $0 \in \phi(B_\epsilon(0)).$

Combinatorially equivalent

$\rightarrow \phi: P \to Q$ is surjective.
The Wachspress map $\phi: P \to Q$

$$\phi(x) := \sum_i \alpha_i(x) q_i.$$ 

**Question:** When do we have $\phi(x) = 0$ for some $x \in \text{int}(P)$?

**Centrally symmetric**

$\rightarrow \phi(0) = 0.$

**Small perturbations**

$\rightarrow$ if $0 \in B_\epsilon(0) \subset P$, then $0 \in \phi(B_\epsilon(0))$.

**Combinatorially equivalent**

$\rightarrow \phi: P \to Q$ is surjective. **Conjecture (Floater):** $\phi$ is injective.
Conjectures
Conjecture.

A polytope is determined (up to isometry) by its edge-graph, edge lengths and the distance of each vertex from some common interior point.

Conjecture.

The edge-graph and edge lengths determine the combinatorial type.

Conjecture. (strengthening Cauchy's rigidity theorem)

A polytope is uniquely determined by its 2-skeleton and the shape of its 2-faces.
Thank you.