Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints

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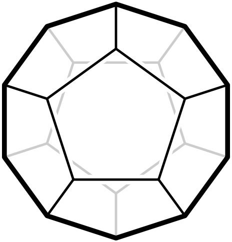
31. May, 2023

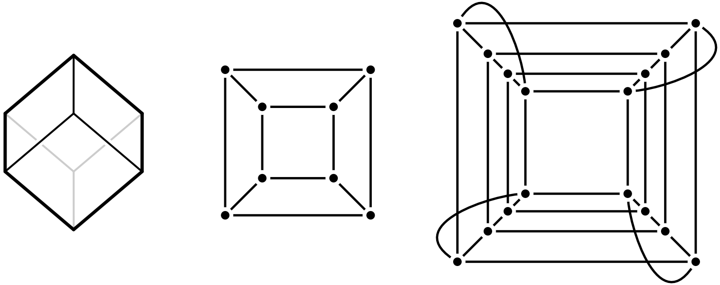
The setting: convex polytopes

$$P = \mathrm{conv}\{p_1, ..., p_n\} \subset \mathbb{R}^d$$



- always convex!
- general dimension $d \geq 2$
- general geometry & combinatorics (not always simple/simplicial/lattice/...)
- always of full dimension

Combinatorics of polytopes

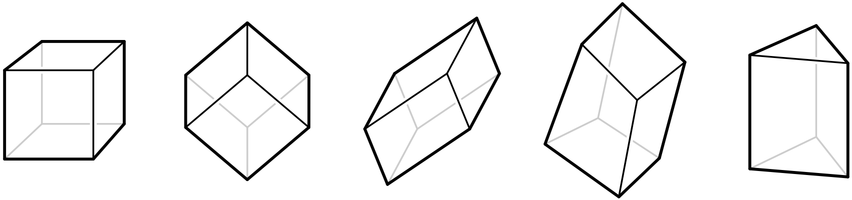


edge-graph ... $G_P := \{$ vertices and edges of $P \}$

face lattice ... $\mathcal{F}(P) := \{$ faces of P ordered by inclusion $\}$

Reconstruction of polytopes

"In how far is a polytope determined by partial combinatorial and geometric data, up to isometry, affine transformation or combinatorial equivalence?"



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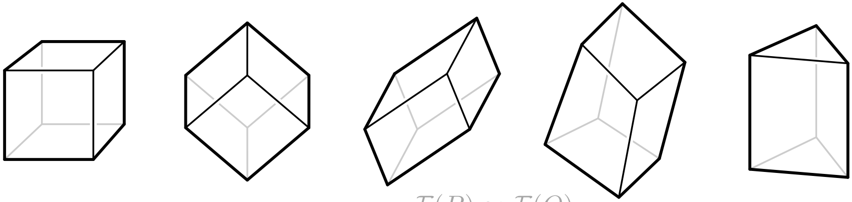

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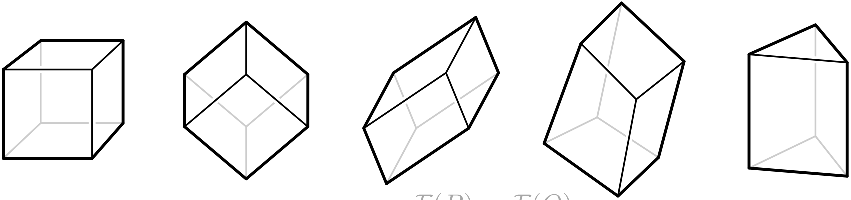


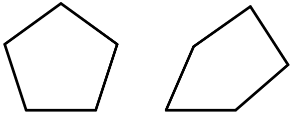
$\mathcal{F}(P) \simeq \mathcal{F}(Q)$

partial data $\longrightarrow$ combinatorics $\longrightarrow$ geometry

Example: edge-graph & edge lengths

Simple polytopes:

- combinatorics can be reconstructed (Blind & Mani; Kalai)
- geometry cannot be reconstructed

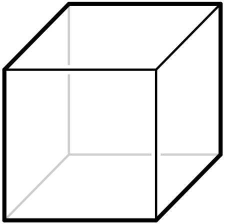


Simplicial polytopes:

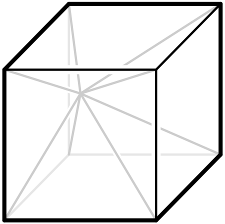
- geometry can be reconstructed, once combinatorics in known (Cauchy)
- combinatorics cannot always be reconstructed (cyclic polytopes)

... *what additional data is needed to permit a reconstruction?*

Two topics



flexible polytopes

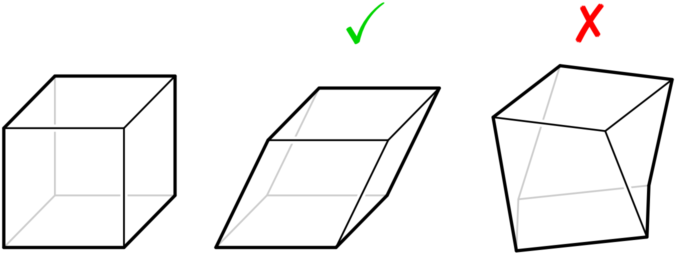


pointed polytopes

Flexible polytopes

(with Bernd Schulze)

Flexing a polytope



- preserving edge lengths

    but also

- preserve planarity of faces
- preserve convexity
- preserve combinatorial type

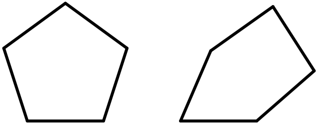
Flexing a polytope

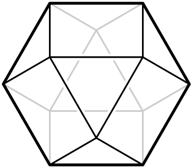
Examples of *rigid* polytopes:

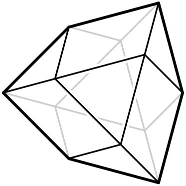
- every simplicial polytope (by Cauchy's rigidity theorem)
- every polytope with triangular 2-faces (e.g. 24-cell)

Examples of *flexible* polytopes:

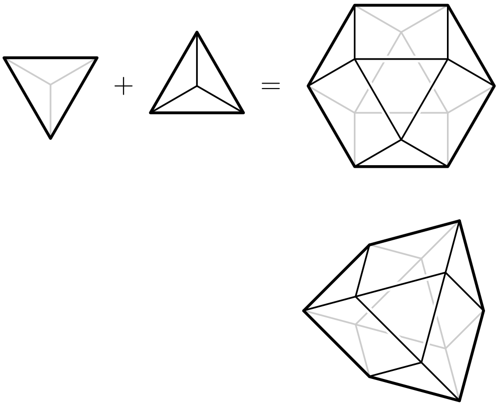
- polygons
- cubes and other prisms







Minkowski sums $A + B := \{a + b \mid a \in A, b \in B\}$



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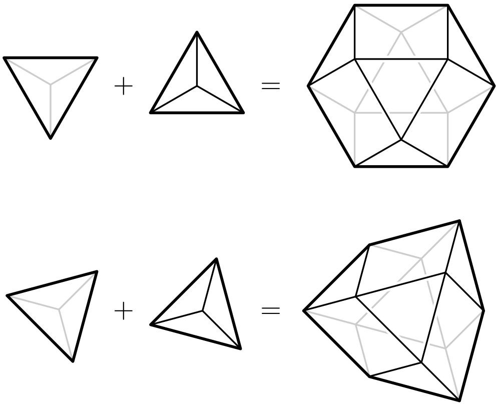

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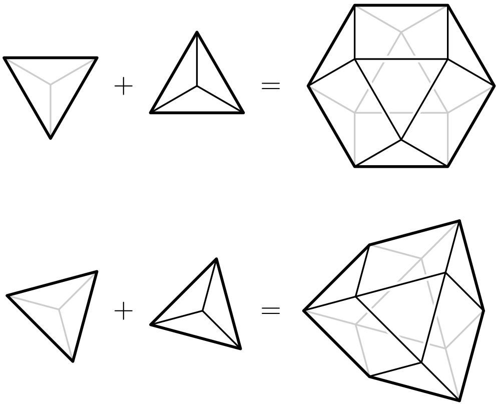
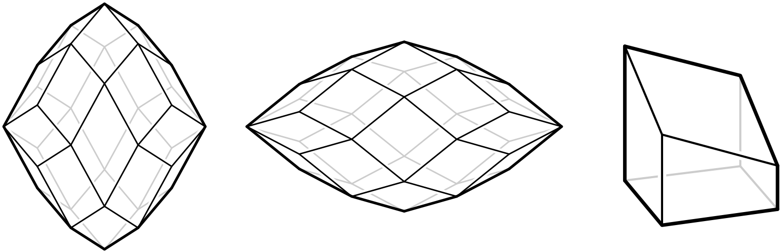


This includes all **zonotopes** $:=$ Minkowski sums of line segments

Affine flexes := flexes realized by affine transformations



Theorem. (Connelly, Gortler, Theran, 2018)

A framework has an affine flex $\iff$ *its edge directions lie on a conic at* ∞.

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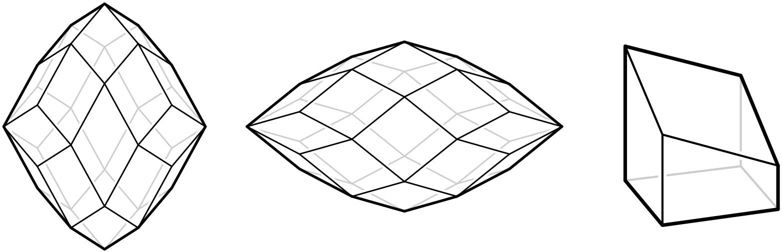

Theorem. (Connelly, Gortler, Theran, 2018)

A framework has an affine flex $\iff$ *its edge directions lie on a conic at* ∞.

For example, if there are at most 5 edge directions (in 3D).

Summarizing ...

We know the following classes of flexible polytopes

- polygons
- Minkowski sums
- all edges on a conic at ∞ (e.g. at most five edge directions)

Question: Are there any others?

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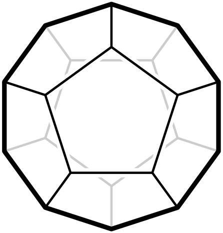
Question: Are there any others?

Some other facts:

- $d = 3$: DOF minus constraints $= 0$
- for all known cases, flexibility is preserved under affine transformations

Toy example: the regular dodecahedron



Question: Is the regular dodecahedron rigid?

- probably, but *we don't know*
- 5-dimensional space of non-trivial infinitesimal flexes
- infinitesimal flexes vanish for any linear transformation (that I tried)

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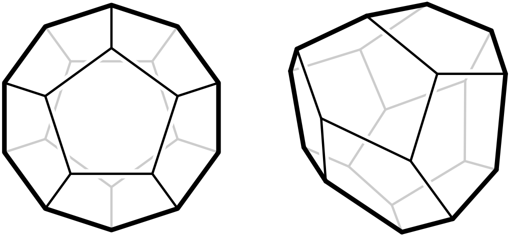

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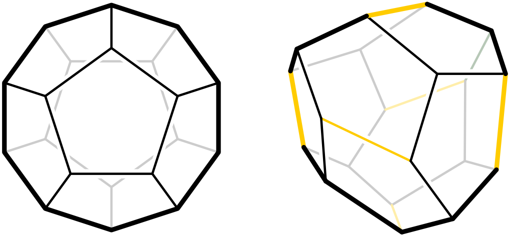

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Is the dodecahedron flexible (as a polytope with fixed edge-lengths)?

Asked 5 months ago Modified 5 months ago Viewed 317 times

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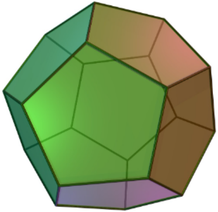
Consider the (regular) dodecahedron $D \subset \mathbb{R}^3$. I want to continuously deform it so that throughout the deformation

1. it stays a convex polytope,
2. it stays a combinatorial dodecahedron (i.e. its edge-graph does not change), and
3. all edge lengths stay the same.

Can I do this? If No, can I do it for some other realizations of the dodecahedron that is not necessarily regular? If Yes, is this possible for *all* realizations?



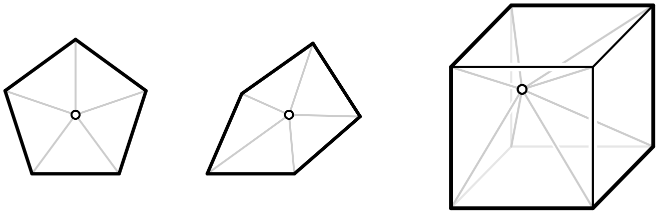
A physical model ...

Rigidity of pointed polytopes

"Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints"
(arXiv:2302.14194)

Pointed polytopes

$:=$ polytope $P \subset \mathbb{R}^d$ + point $x_P \in \mathbb{R}^d$



Conjecture. (W., 2023)

If $x_P \in \operatorname{int}(P)$, then a pointed polytope is uniquely determined (up to isometry) by its edge-graph, edge lengths and vertex-point distances.

implies *e.g.* reconstruction of matroids from base exchange graph

Point in the interior is necessary ...

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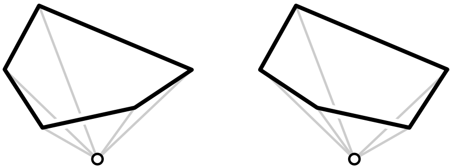


Tensegrity version

Conjecture. (W., 2023)

If $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ are pointed polytopes with the same edge-graph and

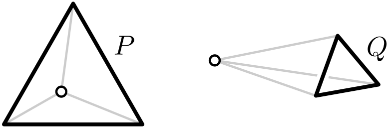
(i) $x_Q \in \mathrm{int}(Q)$

(ii) *edges in Q are at most as long as in P,*

(iii) *vertex-point distances in Q are at least as large as in P,*

then P and Q are isometric.

"A polytope cannot become larger if all its edges become shorter."



MAIN RESULT (W., 2023)

The conjecture holds in the following cases:

I. Q is a small perturbation of P

- ▶ one can replace Q by a graph embedding $q\colon G_P \to \mathbb{R}^d$
- $\cong$ locally rigid as a framework (actually *prestress stable*)

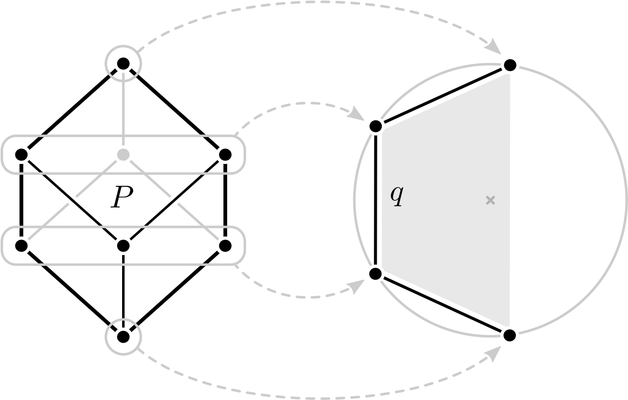
II. P and Q are combinatorially equivalent

- $\cong$ globally rigid as a polytope

III. P and Q are centrally symmetric

- ▶ one can replace Q by a centrally symmetric graph embedding $q\colon G_P \to \mathbb{R}^e$
- $\cong$ universally rigid as a framework

Not globally rigid as a framework



Warmup: simplices

$P, Q \subset \mathbb{R}^d$ simplices,

(i) $0 \in \mathrm{int}(Q)$,

(ii) edges in Q are at most as long as in P.

(iii) vertex-origin distances Q are at least as large as in P.

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Proof.

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Therefore $P \simeq Q$. □

Expansion of polytopes

Fix $\alpha \in \Delta_n := \{(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n_{\geq 0} \mid \alpha_1 + \cdots + \alpha_n = 1\}$

α-**expansion:** $$\|P\|_\alpha^2 := \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|p_i - p_j\|^2$$

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Theorem. (W., 2023)

Let $P \subset \mathbb{R}^d$ be a polytope and $q\colon G_P \to \mathbb{R}^e$ an embedding of its edge-graph with edges at most as long as in P. If $\alpha \in \Delta_n$ are Wachspress coordinates of some interior point of P, then

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Equality holds if and only if q is an affine transformation of the skeleton of P and has the same edge lengths.

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convex geometry + spectral graph theory

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$$P^\circ(\mathbf{c}) := \{x \in \mathbb{R}^d \mid \langle x, p_i \rangle \le c_i \text{ for all } i \in V(G_P)\}.$$

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$$P^\circ(\mathbf{c}) := \{x \in \mathbb{R}^d \mid \langle x, p_i \rangle \le c_i \text{ for all } i \in V(G_P)\}.$$

Expand $\mathrm{vol}(P^\circ(\mathbf{c}))$ at $\mathbf{c} = \mathbf{1}$:

$$\mathrm{vol}(P^\circ(\mathbf{c})) = \mathrm{vol}(P^\circ) + \langle \tilde\alpha, \mathbf{c} - \mathbf{1} \rangle + \tfrac{1}{2}(\mathbf{c} - \mathbf{1})^\top \tilde M (\mathbf{c} - \mathbf{1}) + \cdots$$

↑ Wachspress coordinates

↑ Izmestiev matrix

Recalling the statement

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$$\begin{aligned} \max \quad & \|q\|_\alpha \\ \text{s.t.} \quad & \|q_i - q_j\| \leq \|p_i - p_j\|, \quad \text{for all } ij \in E \\ & q_1, \dots, q_n \in \mathbb{R}^n \end{aligned}$$

Proof: via semidefinite programming

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PROOF: VIA SEMIDEFINITE PROGRAMMING

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$$\begin{aligned} \min \quad & \textstyle\sum_{ij \in E} w_{ij} \|p_i - p_j\|^2 \\ \text{s.t.} \quad & L_w - \operatorname{diag}(\alpha) + \mu \alpha \alpha^\top \succeq 0 \\ & w \ge 0, \mu \text{ free} \end{aligned}$$

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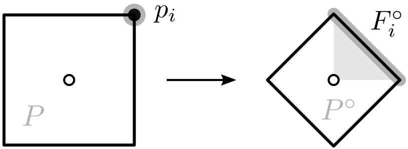
Consequences

Corollary.

A polytope is uniquely determined (up to affine transformations) by its edge-graph, edge lengths and the Wachspress coordinates of any interior point.

$$\alpha_i = \frac{\mathrm{vol}(F_i)}{\|p_i\| \,\mathrm{vol}(P)}$$



Are we done ... ?

$$\sum_i \alpha_i \|p_i\|^2 = \Big\| \sum_i \alpha_i p_i \Big\|^2 + \|P\|_\alpha$$

$$\wedge| \qquad\qquad \vee| \qquad\qquad \vee|$$

$$\sum_i \alpha_i \|q_i\|^2 = \Big\| \sum_i \alpha_i q_i \Big\|^2 + \|Q\|_\alpha$$

Theorem. (W., 2023)

Let $P \subset \mathbb{R}^d$ be a polytope and $q\colon G_P \to \mathbb{R}^e$ an embedding of its edge-graph with edges at most as long as in P. If $\alpha \in \Delta_n$ are Wachspress coordinates of some interior point of P, then

$$\|P\|_\alpha \ge \|q\|_\alpha.$$

Are we done ... ?

$$\sum_i \alpha_i q_i = 0$$

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The Wachspress map $\phi\colon P \to Q$

$$\phi(x) := \sum_i \alpha_i(x) q_i.$$

Question: When do we have $\phi(x) = 0$ for some $x \in \operatorname{int}(P)$?

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Centrally symmetric

→ $\phi(0) = 0$.

Small perturbations

→ if $0 \in B_\epsilon(0) \subset P$, then $0 \in \phi(B_\epsilon(0))$.

Combinatorially equivalent

→ $\phi\colon P \to Q$ is surjective.

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Combinatorially equivalent

→ $\phi\colon P \to Q$ is surjective. **Conjecture (Floater):** ϕ is injective.

CONJECTURES

Conjectures

Conjecture.

A polytope is determined (up to isometry) by its edge-graph, edge lengths and the distance of each vertex from some common interior point.

Conjecture.

The edge-graph and edge lengths determine the combinatorial type.

Conjecture. (strengthening Cauchy's rigidity theorem)

A polytope is uniquely determined by its 2-skeleton and the shape of its 2-faces.

Thank you.