A collection of sums over Catalan paths

and its applications to some parking function enumeration problems

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Outline

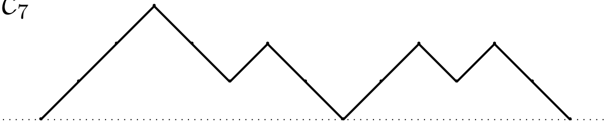
Catalan paths

Definition

- A Catalan path of length $2n$ is a sequence of n up-steps and n down-steps that never goes below the y-axis.
- The set of Catalan paths of length $2n$ is denoted by $\mathcal{C}_n$.
- For every Catalan path C, let $\mathbf{u}(C)$ be the vector recording the lengths of maximum blocks of consecutive up-steps in C.

A Catalan path $C \in \mathcal{C}_7$

$$\mathbf{u}(C) = (3, 1, 2, 1)$$



Sums over Catalan paths

In this talk, we consider sums of the following form:

$$\sum_{C\in\mathcal{C}_n}\prod_{i=1}^{|\mathbf{u}(C)|}\mathbf{u}(C)_i,\quad \sum_{C\in\mathcal{C}_n}\prod_{i=1}^{|\mathbf{u}(C)|}(\mathbf{u}(C)_i)!,\quad \sum_{C\in\mathcal{C}_n}\prod_{i=1}^{|\mathbf{u}(C)|}(1+m\mathbf{u}(C)_i),$$

$$\sum_{C\in\mathcal{C}_n}\prod_{i=1}^{|\mathbf{u}(C)|-1}(1+m\mathbf{u}(C)_i),\quad \sum_{C\in\mathcal{C}_n}\prod_{i=2}^{|\mathbf{u}(C)|}(1+m\mathbf{u}(C)_i),$$

$$\sum_{C\in\mathcal{C}_n}\prod_{i=2}^{|\mathbf{u}(C)|}\left(1+m\sum_{j=i}^{|\mathbf{u}(C)|}\mathbf{u}(C)_j\right).$$

Outline

Canonical decompositions of Catalan paths

Definition

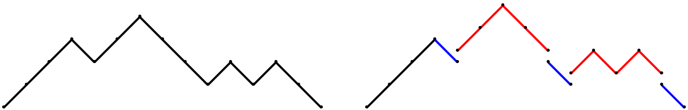
For each $C \in \mathcal{C}_n$, it is easy to see that C has a unique decomposition of the form

$$C = U_1 \cdots U_k D_1 C_1 \cdots D_k C_k,$$

where $U_1, \cdots, U_k$ are the first k consecutive up-steps in C, $D_1, \cdots, D_k$ are down-steps, and $C_1, \cdots, C_k$ are themselves Catalan paths, possibly of length 0. Moreover, this decomposition is reversible.

We call this the canonical decomposition of C.

Note that $\mathbf{u}(C) = (k, \mathbf{u}(C_1), \cdots, \mathbf{u}(C_k))$.



Evaluations using canonical decomposition

Theorem (Y., 2024+)

$$p_n = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|} \mathbf{u}(C)_i = \frac{1}{n+1} \sum_{k=1}^{n} \binom{n+1}{k} \binom{n+k-1}{2k-1}$$

Proof.

From the canonical decomposition $C = U_1 \cdots U_k D_1 C_1 \cdots D_k C_k$, it follows that the generating function $P(x) = 1 + \sum_{n \geq 1} p_n x^n$ satisfies

$$P(x) = 1 + \sum_{k \geq 1} k x^k P(x)^k = 1 + \frac{xP(x)}{(1 - xP(x))^2}.$$

We can now apply Lagrange's Implicit Function Theorem to get a formula for the coefficients p_n, without having to solve for $P(x)$ explicitly. □

Evaluations using canonical decomposition

We can use the same method to obtain the following evaluations.

Theorem (Y., 2024+)

$$\sum_{C\in\mathcal{C}_n}\prod_{i=1}^{|\mathbf{u}(C)|}(\mathbf{u}(C)_i)! = \frac{1}{n+1}\sum_{\substack{a_1+\cdots+a_{n+1}=n\\ a_1,\cdots,a_{n+1}\geq 0}} a_1!\cdots a_{n+1}!, \tag{1}$$

$$\sum_{C\in\mathcal{C}_n}\prod_{i=1}^{|\mathbf{u}(C)|}(1+m\mathbf{u}(C)_i) = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\binom{3n+1-k}{2n+1}(m-1)^k.$$

In general, all sums of the following type can be transformed into something like (1). Depending on f, further simplifications might be possible.

$$\sum_{C\in\mathcal{C}_n}\prod_{i=1}^{|\mathbf{u}(C)|} f(\mathbf{u}(C)_i).$$

Evaluation of variants using canonical decomposition

Theorem (Y., 2024+)

$$p_n = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|-1} (1 + \mathbf{u}(C)_i) = \frac{\binom{3n+1}{n}}{n+1} - \sum_{k=0}^{n-1} \frac{\binom{3n-3k+1}{n-k}}{2^{k+1}(n-k+1)}.$$

Proof sketch.

Let $P(x)$ be the generating function of p_n.
Let $Q(x)$ be the g.f. of $q_n = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|} (1 + \mathbf{u}(C)_i)$, which can be computed as before.
Then, using the canonical decomposition we get

$$P(x) = \sum_{k \geq 1} x^k + \sum_{k \geq 1} (k+1) x^k \sum_{j=1}^{k} Q(x)^{j-1} P(x).$$

A trickier sum

Theorem (Y., 2024+)

Fix $m \geq 1$ *and let*

$$p_n = \sum_{C \in \mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + m \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j \right).$$

Then, $p_n = \sum_{k=1}^{n} p_{n,k}$, *where*

$$p_{n,k} = \begin{cases} 1, & \text{if } k = n, \\ (1 + m(n-k)) \sum_{i=n-k}^{n-1} \sum_{j=k+1-n+i}^{i} p_{i,j}, & \text{if } 1 \leq k \leq n-1. \end{cases}$$

A trickier sum

Theorem (Y., 2024+)

Fix $m \geq 1$ and let

$$p_n = \sum_{C \in \mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + m \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j \right).$$

Then the sequence p_n satisfies

$$\frac{x}{1-x} = \sum_{n=1}^{\infty} p_n \frac{x^n (1-x)^n}{\prod_{\ell=1}^{n} (1 + m\ell x)}.$$

Question (Open)

Is there a more explicit formula for either p_n or its generating function?

Outline

Parking functions

One by one, n cars enter a one-way parking lot with n parking spots.

For each $i \in [n]$, the i-th car drives straight to the $f(i)$-th parking spot, and parks there if it is still available.

Otherwise, it continues down the parking lot and parks at the first available spot, or exits without parking if there isn't one.

Definition

A function $f : [n] \to [n]$ is a parking function if all n cars park successfully.

An example parking function

i	1	2	3	4	5	6
$f(i)$	4	2	4	5	2	1

				1		
$\rightarrow$		2		1		
$\rightarrow$		2		1	3	
$\rightarrow$		2		1	3	4
$\rightarrow$		2	5	1	3	4
$\rightarrow$	6	2	5	1	3	4

Outline

1. A collection of sums over Catalan paths
   - The sums
   - Evaluations
2. Applications to parking function enumeration problems
   - Parking functions
   - Application 1: pattern avoidance in parking functions
   - More applications

Application 1: pattern avoidance in parking functions

Definition

A parking function $f : [n] \to [n]$ avoids pattern σ if its final parking position ρ_f, viewed as a permutation in S_n, avoids σ.

i	1	2	3	4	5	6
$f(i)$	4	2	4	5	2	1

6	2	5	1	3	4

$$\rho_f = 625134$$

Definition

Let $\mathrm{pk}_n(\sigma_1, \cdots, \sigma_k)$ be the number of parking functions $f : [n] \to [n]$ avoiding all of $\sigma_1, \cdots, \sigma_k$.

Example: $\mathrm{pk}_n(123)$

Theorem (Y., 2024+)

$$\mathrm{pk}_n(123) = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|} \mathbf{u}(C)_i = \frac{1}{n+1} \sum_{k=1}^{n} \binom{n+1}{k} \binom{n+k-1}{2k-1}.$$

Lemma

For any $\rho \in S_n$ and $i \in [n]$, let

$$\ell(i,\rho) = \max\{\ell \mid \rho(j) \le \rho(i) \text{ for all } i-\ell+1 \le j \le i\},$$

$$\ell(\rho) = \prod_{i=1}^{n} \ell(i,\rho).$$

Then, $\ell(\rho)$ is the number of parking function $f : [n] \to [n]$ with $\rho_f = \rho$.

Example: $\mathrm{pk}_n(123)$

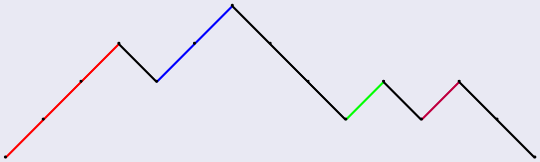
Proof sketch.

- From the lemma, we have

$$\mathrm{pk}_n(123) = \sum_{\substack{\rho\in S_n \\ \rho \text{ avoids } 123}} \ell(\rho) = \sum_{\substack{\rho\in S_n \\ \rho \text{ avoids } 123}} \prod_{i=1}^{n} \ell(i,\rho).$$

- There is a bijection mapping every $\rho \in S_n$ avoiding 123 to a Catalan path C of length $2n$, such that $\ell(\rho) = \prod_{i=1}^{|\mathbf{u}(C)|} \mathbf{u}(C)_i$.

$\rho = 5471632$



More results

Using similar bijections to $\mathcal{C}_n$, and determining the correspondence between $\ell(\rho)$ and $\mathbf{u}(C)$, we obtain the following.

Theorem (Y., 2024+)

$$\mathrm{pk}_n(213) = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|} (\mathbf{u}(C)_i)! = \frac{1}{n+1} \sum_{\substack{a_1+\cdots+a_{n+1}=n \\ a_1,\cdots,a_{n+1}\geq 0}} a_1! \cdots a_{n+1}!,$$

$$\mathrm{pk}_n(312) = \sum_{C \in \mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j\right),$$

$$\mathrm{pk}_n(321) = \sum_{C \in \mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j\right) \prod_{i=1}^{|\mathbf{u}(C)|} (\mathbf{u}(C)_i - 1)!.$$

Outline

Application 2: another notion of pattern avoidance

Under a different notion of pattern avoidance in parking functions, Adeniran and Pudwell obtained the following.

Theorem (Adeniran, Pudwell, 2023)

$$\mathrm{pf}_n(312, 321) = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|-1} (1 + \mathbf{u}(C)_i).$$

Using the canonical decomposition, we can evaluate this sum as

Theorem (Y., 2024+)

$$\mathrm{pf}_n(312, 321) = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|-1} (1 + \mathbf{u}(C)_i) = \frac{\binom{3n+1}{n}}{n+1} - \sum_{k=0}^{n-1} \frac{\binom{3n-3k+1}{n-k}}{2^{k+1}(n-k+1)}.$$

Application 3: Hopf algebras of generalised parking functions

These sums can also be applied to Novelli and Thibon's work on Hopf algebras of generalised parking functions to compute their dimensions, which are equal to the number of congruence classes.

Theorem (Novelli, Thibon, 2020 & Y., 2024+)

The number of hyposylvester classes of m-multiparking functions is

$$\sum_{C \in \mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} (1 + m\mathbf{u}(C)_i) = \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} \binom{3n-k}{2n+1} (m-1)^k.$$

The number of metasylvester classes of m-multiparking functions is

$$\sum_{C \in \mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + m \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j \right).$$

Application 3: Hopf algebras of generalised parking functions

Let $\mathcal{C}_n^{(m)}$ be the set of m-Catalan paths and define $\mathbf{u}(C)$ analogously.

Theorem (Novelli, Thibon, 2020 & Y., 2024+)

The number of hyposylvester classes of m-parking functions is

$$\sum_{C \in \mathcal{C}_n^{(m)}} \prod_{i=2}^{|\mathbf{u}(C)|} (1 + \mathbf{u}(C)_i) = \frac{1}{2mn+1} \binom{(2m+1)n}{n}.$$

The number of metasylvester classes of m-parking functions is

$$\sum_{C \in \mathcal{C}_n^{(m)}} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j \right). \tag{2}$$

Notably, unlike its $\mathcal{C}_n$ counterpart, we haven't been able to obtain even a nice recurrence formula for (2).

Open problems and further directions

- Are there more explicit formulas for sums, or their generating functions, of the following type?

$$\sum_{C\in\mathcal{C}_n}\prod_{i=2}^{|\mathbf{u}(C)|}\left(1+\sum_{j=i}^{|\mathbf{u}(C)|}\mathbf{u}(C)_j\right).$$

- Investigate analogous sums over generalised Catalan paths, Schröder paths or Motzkin paths, or with different summands. Are they connected to other enumeration problems on parking functions or other objects?