# A collection of sums over Catalan paths

## and its applications to some parking function enumeration problems

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# Outline

1. A collection of sums over Catalan paths
   - The sums
   - Evaluations
2. Applications to parking function enumeration problems
   - Pattern avoidance in parking functions
   - Another notion of pattern avoidance
   - Hopf algebras of generalised parking functions

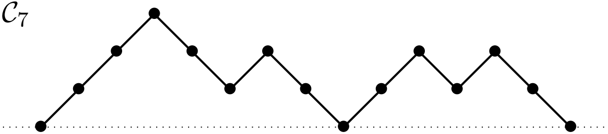
# Catalan paths

**Definition**

- A Catalan path of length $2n$ is a sequence of $n$ up-steps and $n$ down-steps that never goes below the $y$-axis.
- The set of Catalan paths of length $2n$ is denoted by $\mathcal{C}_n$.
- For every Catalan path $C$, let $\mathbf{u}(C)$ be the vector recording the lengths of maximum blocks of consecutive up-steps in $C$.

A Catalan path $C \in \mathcal{C}_7$

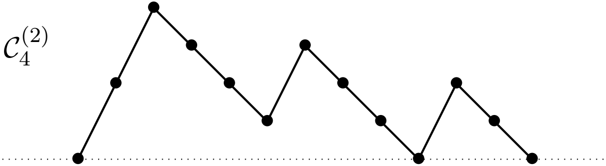
$$\mathbf{u}(C) = (3, 1, 2, 1)$$



# Other path families

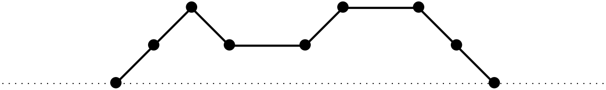
A 2-Catalan path $C \in \mathcal{C}_4^{(2)}$

$$\mathbf{u}(C) = (2, 1, 1)$$



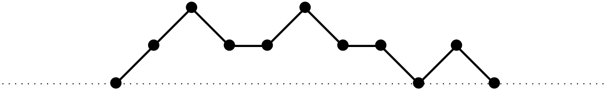
A Schröder path $S \in \mathcal{S}_5$

$$\mathbf{u}(S) = (2, 1)$$



A Motzkin path $M \in \mathcal{M}_{10}$

$$\mathbf{u}(M) = (2, 1, 1)$$



# Sums over Catalan paths

In this talk, we mainly consider sums of the form

$$\sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|} f(\mathbf{u}(C)_i),$$

where $f : \mathbb{Z}^+ \to \mathbb{C}$ is some weight function, and their variants

$$\sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|-1} f(\mathbf{u}(C)_i), \qquad \sum_{C \in \mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} f(\mathbf{u}(C)_i).$$

# Canonical decompositions of Catalan paths

## Definition

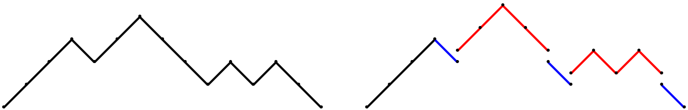
For each $C \in \mathcal{C}_n$, it is easy to show that $C$ has a unique decomposition of the form

$$C = U_1 \cdots U_\ell D_1 C_1 \cdots D_\ell C_\ell,$$

where $U_1, \cdots, U_\ell$ are the first $\ell$ consecutive up-steps in $C$, $D_1, \cdots, D_\ell$ are down-steps, and $C_1, \cdots, C_\ell$ are themselves Catalan paths, possibly of length 0. Moreover, this decomposition is reversible.

We call this the canonical decomposition of $C$.

Note that $\mathbf{u}(C) = (\ell, \mathbf{u}(C_1), \cdots, \mathbf{u}(C_\ell))$.



# Evaluations using the canonical decomposition

## Theorem (Y., 2024+)

*For any function* $f : \mathbb{Z}^+ \to \mathbb{C}$, *let* $F(x) = 1 + \sum_{n\geq 1} f(n)x^n$. *Then*

$$p_n = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|} f(\mathbf{u}(C)_i) = \frac{1}{n+1}[x^n]F(x)^{n+1}$$

## Proof.

From the canonical decomposition $C = U_1 \cdots U_\ell D_1 C_1 \cdots D_\ell C_\ell$, it follows that the generating function $P(x) = 1 + \sum_{n\geq 1} p_n x^n$ satisfies

$$P(x) = 1 + \sum_{\ell \geq 1} f(\ell)x^\ell P(x)^\ell = F(xP(x)).$$

The result then follows from Lagrange Inversion. □

# Evaluations using the canonical decomposition

We can use analogous decompositions to sums over other path families

**Theorem (Y., 2024+)**

$$\sum_{C\in\mathcal{C}_n^{(k)}} \prod_{i=1}^{|\mathbf{u}(C)|} f(\mathbf{u}(C)_i) = \frac{1}{kn+1}[x^n]F(x)^{kn+1},$$

$$\sum_{S\in\mathcal{S}_n} \prod_{i=1}^{|\mathbf{u}(S)|} f(\mathbf{u}(S)_i) = \frac{1}{n+1}[x^n](1+x)^{n+1}F(x)^{n+1},$$

$$\sum_{M\in\mathcal{M}_n} \prod_{i=1}^{|\mathbf{u}(M)|} f(\mathbf{u}(M)_i) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{n-i+1}\binom{n-i+1}{i+1}[t^i]F(t)^{n-i+1}.$$

As well as the variants

$$\sum_{C\in\mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|-1} f(\mathbf{u}(C)_i), \qquad \sum_{C\in\mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} f(\mathbf{u}(C)_i).$$

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# Application 1: pattern avoidance in parking functions

## Definition

A parking function $f : [n] \to [n]$ avoids pattern $\sigma$ if its final parking position $\rho_f$, viewed as a permutation in $S_n$, avoids $\sigma$.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $f(i)$ | 4 | 2 | 4 | 5 | 2 | 1 |

| | | | | | |
|---|---|---|---|---|---|
| 6 | 2 | 5 | 1 | 3 | 4 |

$\rho_f = 625134$

## Definition

Let $\mathrm{pk}_n(\sigma_1, \cdots, \sigma_k)$ be the number of parking functions $f : [n] \to [n]$ avoiding all of $\sigma_1, \cdots, \sigma_k$.

# Application 1: pattern avoidance in parking functions

Using bijections between permutations in $S_n$ avoiding the pattern 123 (or 213) and Catalan paths in $\mathcal{C}_n$, and the general results earlier, we obtain

**Theorem (Y., 2024+)**

$$\mathrm{pk}_n(123) = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|} \mathbf{u}(C)_i = \frac{1}{n+1} \sum_{k=1}^{n} \binom{n+1}{k} \binom{n+k-1}{2k-1}.$$

$$\mathrm{pk}_n(213) = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|} (\mathbf{u}(C)_i)! = \frac{1}{n+1} \sum_{\substack{a_1+\cdots+a_{n+1}=n \\ a_1,\cdots,a_{n+1}\geq 0}} a_1! \cdots a_{n+1}!,$$

# Application 1: pattern avoidance in parking functions

Using similar bijections, we can also obtain

**Theorem (Y., 2024+)**

$$\mathrm{pk}_n(312) = \sum_{C \in \mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} \left( 1 + \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j \right),$$

$$\mathrm{pk}_n(321) = \sum_{C \in \mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} \left( 1 + \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j \right) \prod_{i=1}^{|\mathbf{u}(C)|} (\mathbf{u}(C)_i - 1)!.$$

The canonical decomposition method does not work for these sums. But we can still obtain a recurrence formula using a different decomposition method.

# Application 2: another notion of pattern avoidance

Under a different notion of pattern avoidance in parking functions, Adeniran and Pudwell obtained the following.

**Theorem (Adeniran, Pudwell, 2023)**

$$\mathrm{pf}_n(312, 321) = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|-1} (1 + \mathbf{u}(C)_i).$$

Using the canonical decomposition, we can evaluate this sum as

**Theorem (Y., 2024+)**

$$\mathrm{pf}_n(312, 321) = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|-1} (1+\mathbf{u}(C)_i) = \frac{\binom{3n+1}{n}}{n+1} - \sum_{k=0}^{n-1} \frac{\binom{3n-3k+1}{n-k}}{2^{k+1}(n-k+1)}.$$

# Application 3: Hopf algebras of generalised parking functions

These sums can also be applied to Novelli and Thibon's work on Hopf algebras of generalised parking functions to compute their dimensions, which are equal to the number of congruence classes.

**Theorem (Novelli, Thibon, 2020 & Y., 2024+)**

*The number of hyposylvester classes of $m$-multiparking functions is*

$$\sum_{C\in\mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} (1 + m\mathbf{u}(C)_i) = \frac{1}{n}\sum_{k=0}^{n-1} \binom{n}{k}\binom{3n-k}{2n+1}(m-1)^k.$$

*The number of metasylvester classes of $m$-multiparking functions is*

$$\sum_{C\in\mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + m\sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j\right).$$

# Application 3: Hopf algebras of generalised parking functions

**Theorem (Novelli, Thibon, 2020 & Y., 2024+)**

*The number of hyposylvester classes of $m$-parking functions is*

$$\sum_{C \in \mathcal{C}_n^{(m)}} \prod_{i=2}^{|\mathbf{u}(C)|} (1 + \mathbf{u}(C)_i) = \frac{1}{2mn+1} \binom{(2m+1)n}{n}.$$

*The number of metasylvester classes of $m$-parking functions is*

$$\sum_{C \in \mathcal{C}_n^{(m)}} \prod_{i=2}^{|\mathbf{u}(C)|} \left( 1 + \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j \right). \tag{1}$$

Notably, unlike its $\mathcal{C}_n$ counterpart, we haven't been able to obtain even a nice recurrence formula for (1).

# Open problems and further directions

- Are there more explicit formulas for sums, or their generating functions, of the following type?

$$\sum_{C \in \mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} \left( 1 + \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j \right).$$

- Are analogous sums over Schröder and Motzkin paths connected to other enumeration problems?