

An improved hypergraph Mantel's Theorem

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Joint work with Daniel Ilkovič

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Definition

Let \mathcal{F} be a family of r -uniform hypergraphs.

- The **Turán number** $\text{ex}(n, \mathcal{F})$ is the maximum number of edges in an r -uniform hypergraph H on n vertices that does not contain a copy of F for any $F \in \mathcal{F}$.
- The **Turán density** $\pi(\mathcal{F})$ is defined to be

$$\lim_{n \rightarrow \infty} \frac{\text{ex}(n, \mathcal{F})}{\binom{n}{r}}.$$

Turán problem for graphs ($r = 2$)

Very well understood for non-bipartite graphs.

Theorem (Mantel's Theorem, 1907)

- $\pi(K_3) = 1/2$.
- $\text{ex}(n, K_3) = \lfloor n^2/4 \rfloor$, and the unique extremal graph is the balanced complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

Theorem (Erdős-Stone Theorem, 1946)

If $\chi(F) = \chi + 1$, then $\pi(F) = 1 - 1/\chi$ and $\text{ex}(n, F) = (1 - 1/\chi + o(1))\binom{n}{2}$.

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Turán problem for hypergraphs ($r \geq 3$)

Let $T^r(n)$ be the balanced complete r -partite r -uniform hypergraph.

Fact

Let $r \geq 2$ and let \mathcal{F} be a family of r -uniform hypergraphs.

- $\pi(\mathcal{F}) \geq r!/r^r$ if no $F \in \mathcal{F}$ is r -partite, because $T^r(n)$ is \mathcal{F} -free.
- $\pi(\mathcal{F}) = 0$ otherwise.

Question

Can we generalise Mantel's Theorem by finding a family \mathcal{F} of **triangle-like** r -uniform hypergraphs such that $\pi(\mathcal{F}) = r!/r^r$?

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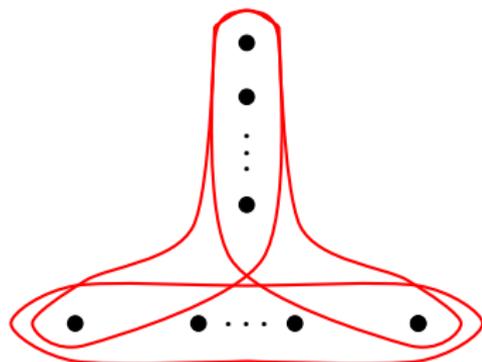
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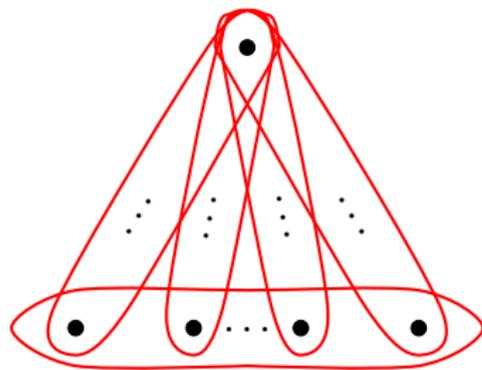
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Some attempts

Let \mathbb{T}_r and $\Delta_{(1,1,\dots,1)}$ be the following r -uniform hypergraphs.



\mathbb{T}_r



$\Delta_{(1,1,\dots,1)}$

Note that $\mathbb{T}_2 = \Delta_{(1,1)} = K_3$.

- [Frankl, Füredi, 1983]
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$$\pi(\mathbb{T}_3) = 3!/3^3.$$

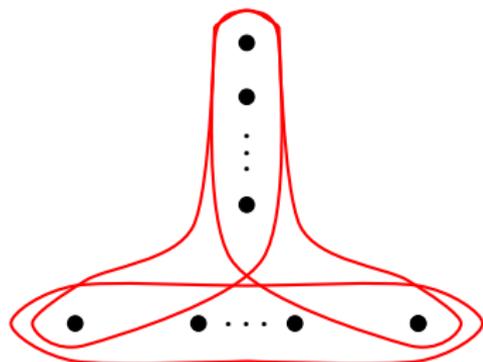
$$\pi(\mathbb{T}_4) = 4!/4^4.$$

$$\pi(\mathbb{T}_5) > 5!/5^5 \text{ and } \pi(\mathbb{T}_6) > 6!/6^6.$$

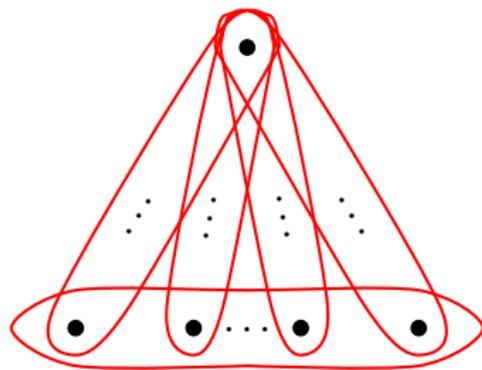
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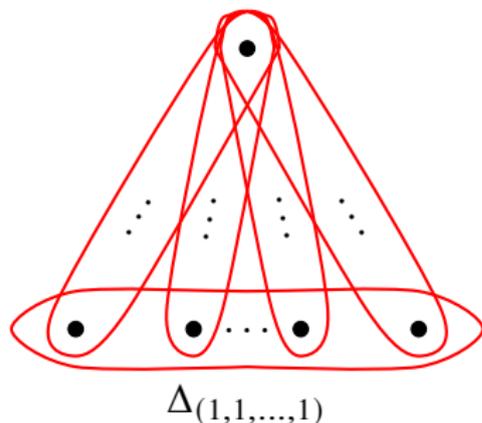
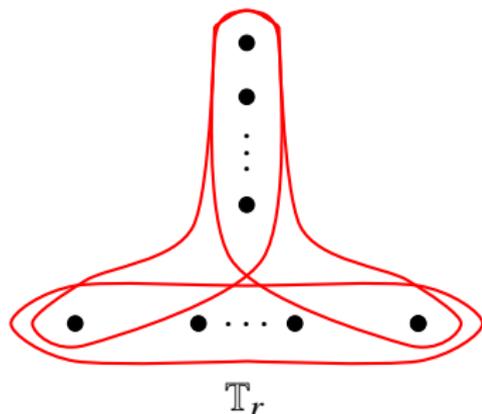
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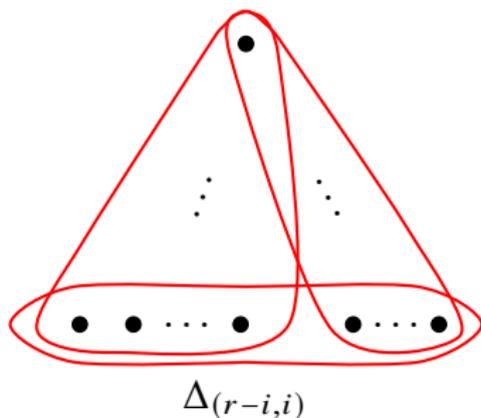
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New attempt

Let $\Delta_{(r-i,i)}$ be given below, and let $\mathcal{F}_{r,k} = \{\Delta_{(r-i,i)} \mid 1 \leq i \leq k\}$.



Theorem (Chao, Yu, 2024+)

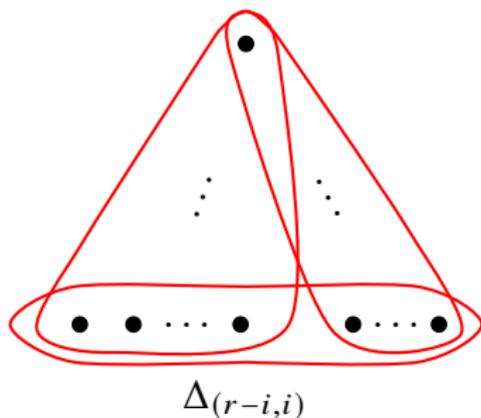
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$\text{ex}(\mathcal{F}_{r, \lfloor r/2 \rfloor}) = |E(T^r(n))|$, with $T^r(n)$ being the unique extremal construction.

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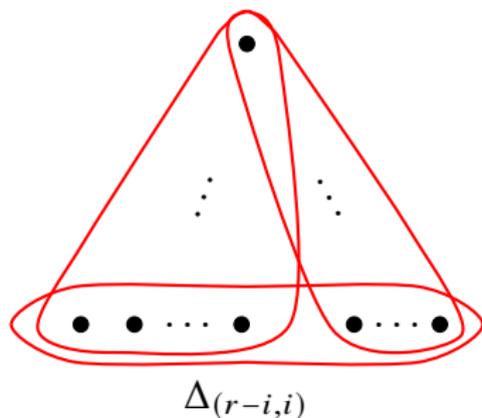
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Our Results

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Theorem (Ilkovič, Y., 2025+)

$$\pi(\mathcal{F}_{r,[r/e]}) = r!/r^r.$$

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Reduction to optimisation with entropy

Chao and Yu's novel entropy method:

Turán density problem

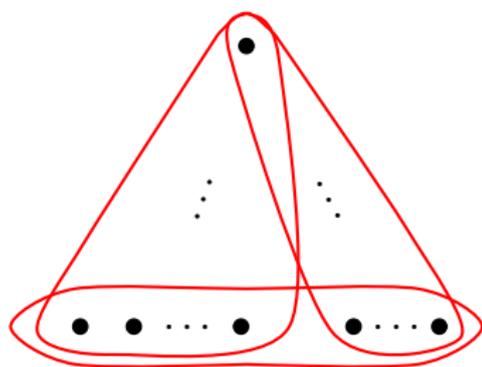


Optimisation problem

$$\pi(\mathcal{F}_{r,k})$$



$$\prod_{i=1}^r x_i$$



$$x_i + x_{r-i} \leq x_r$$

$$x_i + x_j \leq x_{i+j}$$

for each $i \leq j \leq r - i$

$$\Delta(r-i, i)$$

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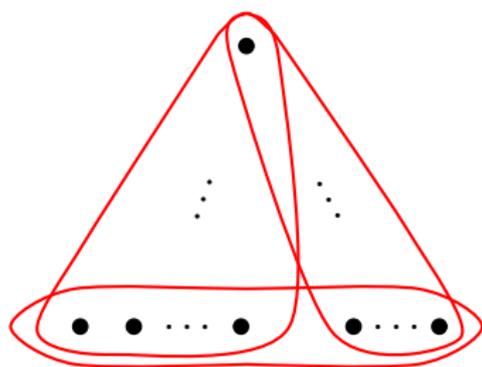


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For every $1 \leq k \leq \lfloor r/2 \rfloor$, define $\mathcal{X}_{r,k} \subset [0, 1]^r$ to be

$$\{0 < x_1 \leq \dots \leq x_r = 1 \mid x_i + x_j \leq x_{i+j} \text{ for every } i \in [k] \text{ and } i \leq j \leq r - i\}.$$

Lemma (Chao, Yu, 2024+)

$$\pi(\mathcal{F}_{r, \lfloor r/2 \rfloor}) \leq \max \left\{ \prod_{i=1}^r x_i \mid (x_1, \dots, x_r) \in \mathcal{X}_{r, \lfloor r/2 \rfloor} \right\}$$

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Moreover, the choice of $\lceil r/e \rceil$ here cannot be improved.

Where does e come from?

Let $\varepsilon > 0$ be sufficiently small, and let $1 \leq k \leq \lfloor r/2 \rfloor$. Consider the sequence (x_1, \dots, x_r) given by

$$x_i = \begin{cases} (1 - \varepsilon)i/r, & \text{if } i \in [k], \\ \varepsilon + (1 - \varepsilon)i/r, & \text{if } k + 1 \leq i \leq r. \end{cases}$$

It is easy to check that $(x_1, \dots, x_r) \in \mathcal{X}_{r,k}$. Define $f(\varepsilon)$ to be

$$f(\varepsilon) = \frac{\prod_{i=1}^r x_i}{r!/r^r} = \frac{\prod_{i=1}^r x_i}{\prod_{i=1}^r \frac{i}{r}} = \prod_{i=1}^k (1 - \varepsilon) \prod_{i=k+1}^r \left(1 + \frac{(r-i)\varepsilon}{i}\right).$$

If $k < \lfloor r/e \rfloor$,

$$f'(0) = -k + \sum_{i=k+1}^r \frac{(r-i)}{i} \geq -k + r \int_{\frac{k+1}{r}}^1 \frac{1-t}{t} dt > 0,$$

so $\prod_{i=1}^r x_i > r!/r^r$ for small ε .

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Theorem (Ilkovič, Y., 2025+)

$$\max \left\{ \prod_{i=1}^r x_i \mid (x_1, \dots, x_r) \in \mathcal{X}_{r, \lceil r/e \rceil} \right\} = \frac{r!}{r^r}.$$

Proof sketch.

From the conditions, $x_1 \leq 1/r$, so suppose $x_1 = (1 - \varepsilon)/r$.

If $x_i = (1 - \varepsilon)i/r$ for all $i \leq \lceil r/e \rceil$, then $x_i \leq \varepsilon + (1 - \varepsilon)i/r$ for all $i > \lceil r/e \rceil$.

Then,

$$\log \left(\frac{\prod_{i=1}^r x_i}{r!/r^r} \right) \leq -k\varepsilon + \sum_{i=k+1}^r \frac{(r-i)\varepsilon}{i} \leq \varepsilon \left(r \int_{\frac{k}{r}}^1 \frac{1-t}{t} dt - k \right) \leq 0.$$

Otherwise, with some **small adjustments** we get $(x'_1, \dots, x'_r) \in \mathcal{X}_{r, \lceil r/e \rceil}$ such that $\prod_{i=1}^r x_i < \prod_{i=1}^r x'_i$, so (x_1, \dots, x_r) is not optimal. \square

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Ma and Zhu (2025+) gave an alternate proof of this with a very clever and sophisticated application of the **AM-GM inequality**.

Recall that the entropy methods cannot prove $\pi(\mathcal{F}_{r,k}) = r!/r^r$ for $k < \lfloor r/e \rfloor$.

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