An interesting forbidden matrix problem

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Outline

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1. Preliminaries
    - The forbidden matrix problem
    - An optimisation problem on multigraphs
2. Relating the two problems
3. Proof sketch
    - Upper bound
    - Lower bound
4. Open questions

Forbidden number forb(m, r, F)

Definition

- An r-matrix is a matrix whose entries all belong to the set $\{0, 1, \cdots, r-1\}$.
- A configuration of a matrix F is a matrix that can be obtained by permuting the rows and columns of F.
- forb(m, r, F) is the maximum number of distinct columns in an m-rowed r-matrix that does not contain a configuration of F.

Example

The 4-matrix $\begin{bmatrix} 2 & 1 & 0 \\ 2 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ contains a configuration of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Forbidden number forb(m, r, F)

Definition

forb(m, r, F) is the maximum number of distinct columns in an m-rowed r-matrix that does not contain a configuration of F.

Example

forb$(m, 2, I_2) = m + 1$ as any two distinct columns with the same number of 1's give rise to a configuration of I_2, while $\begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$ contains no configuration of I_2.

Known results for $r = 2$

The $r = 2$ case has been extensively studied.

- The exact value of $\text{forb}(m, 2, F)$ is known for many small matrices F and many infinite families of matrices F.
- Asymptotic growth of $\text{forb}(m, 2, F)$ is known for many other family of matrices. But there is no complete characterisation yet.

For more information on the $r = 2$ case, see *A survey of forbidden configuration results* by Richard Anstee.

Known results for $r \geq 3$

The $r \geq 3$ case has hardly been explored. We will focus on $\text{forb}(m, r, F)$ in the case when F is a (0,1)-matrix.

Definition

The support of a column c is the set of row indices i satisfying $c_i = 0$ or 1.

Theorem (Dillon, Sali, 2021)

For every (0,1)-matrix F and $r \geq 3$,

$$\text{forb}(m, r, F) \leq \sum_{j=0}^{m} \binom{m}{j} (r-2)^{m-j} \, \text{forb}(j, 2, F).$$

Moreover, equality holds if the sequence of extremal matrices (M_j) attaining $\text{forb}(j, 2, F)$ are "nested".

Known results for $r \geq 3$

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Moreover, equality holds if the sequence of extremal matrices (M_j) attaining $\text{forb}(j, r, F)$ are "nested".

Using this, Dillon and Sali determined $\text{forb}(m, r, F)$ exactly for all 2-rowed and up to 3×3 (0,1)-matrices F with no repeated columns, except

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Main problem and result

Improve the following bounds on $\mathrm{forb}(m, r, M)$ for $r \geq 3$.

$$(r-1)^m + m(r-1)^{m-1} \leq \mathrm{forb}(m, r, M) \leq (r-1)^m + 1.5m(r-1)^{m-1}.$$

$\uparrow$

number of columns
with at most one 0

$\uparrow$

the upper bound theorem and
$\mathrm{forb}(m, 2, M) = \left\lfloor \frac{3m}{2} \right\rfloor + 1$

Theorem (Peaslee, Sali, Y., 2023+)

For all $r \geq 3$,

$$(r-1)^m + 1.360m(r-1)^{m-1} \leq \mathrm{forb}(m, r, M) \leq (r-1)^m + 1.433m(r-1)^{m-1}.$$

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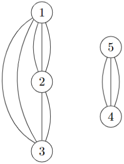
Triangular choice multigraph (TCM)

Definition

A triangular choice multigraph (TCM) $\mathcal{G}$ on a vertex set V is a multigraph obtained by choosing one of edge ij, ik, jk for every unordered triple $i, j, k \in V$, and including it in $\mathcal{G}$.

Example



Vertex Triplet	1,2,3	1,2,4	1,2,5	1,3,4	1,3,5	1,4,5	2,3,4	2,3,5	2,4,5	3,4,5
Subgraph										

An optimisation problem on TCM

Definition

For every TCM $\mathcal{G}$ on $[m]$ and $\alpha \in \mathbb{R}$, let m_{ij} be the multiplicity of ij in $\mathcal{G}$, and let

$$w(\mathcal{G}, \alpha) = \sum_{ij} \alpha^{m_{ij}}.$$

Question

Determine the values of

$$H(m, \alpha) = \max\{w(\mathcal{G}, \alpha) \colon \mathcal{G} \text{ is a TCM on } [m]\}$$

$$H_2(m, \alpha) = \max\{w(\mathcal{G}, \alpha) \colon \mathcal{G} \text{ is a 2-recursive* TCM on } [m]\}$$

*: 2-recursive TCM will be defined later.

Relationship between $\mathrm{forb}(m, r, M)$ and $H(m, \alpha)$, $H_2(m, \alpha)$

Theorem (Peaslee, Sali, Y., 2023+)

For every $r \geq 3$, we have

- $\mathrm{forb}(m, r, M) - (r-1)^m - m(r-1)^{m-1} \leq H(m, \frac{r-1}{r-2})(r-2)^{m-2}$,
- $\mathrm{forb}(m, r, M) - (r-1)^m - m(r-1)^{m-1} \geq H_2(m, \frac{r-1}{r-2})(r-2)^{m-2}$

Theorem (Peaslee, Sali, Y., 2023+)

- $H(m, 2) \leq 0.433 m 2^{m-1}$.
- $H_2(m, \frac{r-1}{r-2})(r-2)^{m-2} \geq 0.360 m(r-1)^{m-1}$

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Choices

Observation

If A contains no configuration of M, then for every triple i, j, k and each pair of columns below, A restricted to rows i, j, k contains at most one column in the pair.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Definition

A choice is a sequence of 3×3 matrices $\mathcal{B} = (B_{i,j,k})$, where for each triple i, j, k, $B_{i,j,k}$ is formed by picking 1 column from each of the 3 pairs above, and putting them together.

forb$(m, r, \mathcal{B})$

Definition

We say A forbids a choice $\mathcal{B}$ if for every triple i, j, k, A restricted to rows i, j, k contains no column in $B_{i,j,k}$.

Observation

Every matrix A that contains no configuration of M forbids a choice $\mathcal{B}$.

Definition

Define forb$(m, r, \mathcal{B})$ to be the maximum number of columns an m-rowed r-matrix A can have if A forbids $\mathcal{B}$.

It follows that

$$\text{forb}(m, r, M) = \max\{\text{forb}(m, r, \mathcal{B}) \colon \mathcal{B} \text{ is a choice}\}.$$

forb$(m, r, \mathcal{B})$ and valid columns

Definition

Let $\mathcal{B}$ be a choice on $[m]$ and let $X \subset [m]$.

- A column c on X is valid with respect to $\mathcal{B}$ if for every triple i, j, k in X, c restricted to rows i, j, k is not a column in $B_{i,j,k}$.
- $c(\mathcal{B}, X)$ is defined to be the number of valid (0,1)-columns on X with respect to $\mathcal{B}$.

Observation

$$\text{forb}(m, r, \mathcal{B}) = \sum_{X \subset [m]} c(\mathcal{B}, X)(r-2)^{m-|X|}.$$

0-implications

Definition

Let $\mathcal{B}$ be a choice on $[m]$. For every $X \subset [m]$ and $i, j \in X$, we say there is a 0-implication from i to j on X if for every valid column c with support X with respect to $\mathcal{B}$, $c_i = 0$ implies $c_j = 0$.

Example

The forbidden conditions imposed by every $B_{i,j,k}$ correspond to 0-implications. In this example, they are represented as arrows.

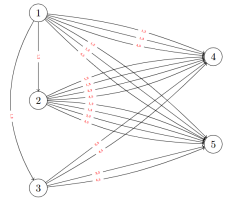
$$\begin{tikzcd}[row sep=small]
 & 0 & 0 & 0 & i \arrow[d, bend left] \arrow[dd, bend left=60] \\
B_{i,j,k} = & 1 & 0 & 1 & j \\
 & 1 & 1 & 0 & k
\end{tikzcd}$$

Associated multigraph $\mathcal{D}_{\mathcal{B}}(X)$

Definition

Given a choice $\mathcal{B}$ on $[m]$ and $X \subset [m]$, the directed multigraph $\mathcal{D}_{\mathcal{B}}(X)$ associated to $\mathcal{B}$ is obtained by drawing all the arrows corresponding to 0-implications imposed by matrices $B_{i,j,k}$.

Example



$B_{1,2,3}$	$B_{1,2,4}$	$B_{1,2,5}$	$B_{1,3,4}$	$B_{1,3,5}$	$B_{1,4,5}$	$B_{2,3,4}$	$B_{2,3,5}$	$B_{2,4,5}$	$B_{3,4,5}$
A_1	A_2	A_2	A_2	A_2	A_1	A_2	A_2	A_1	A_1
									

Number of valid columns on X

Lemma (Peaslee, Sali, Y., 2023+)

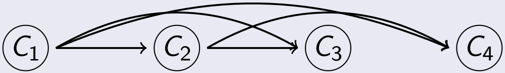
Let $\mathcal{B}$ be a good choice and let $X \subset [m]$, then

$$c(\mathcal{B}, X) \leq n(\mathcal{B}, X) + |X| + 1,$$

where $n(\mathcal{B}, X)$ is the number of unordered pairs of vertices in $\mathcal{D}_{\mathcal{B}}(X)$ with no directed edge between them.

"Proof" by example.

Any valid column is constant on each strongly connected component.



C_1	C_2	C_3	C_4
0	0	0	0
1	0	0	0
1	1	0	0

C_1	C_2	C_3	C_4
1	1	0	1
1	1	1	0
1	1	1	1

The valid columns if $C_1, \cdots, C_4$ are strongly connected components. □

Number of valid columns

Therefore, we have

$$\begin{aligned}\mathrm{forb}(m,r,\mathcal{B}) &= \sum_{X\subset[m]} c(\mathcal{B},X)(r-2)^{m-|X|}\\ &\le \sum_{X\subset[m]} (n(\mathcal{B},X)+|X|+1)(r-2)^{m-|X|}\\ &= (r-1)^m + m(r-1)^{m-1} + \sum_{X\subset[m]} n(\mathcal{B},X)(r-2)^{m-|X|}.\end{aligned}$$

Theorem (Peaslee, Sali, Y., 2023+)

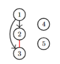
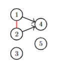
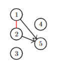
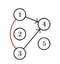
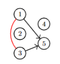
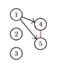
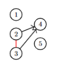
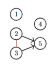
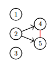
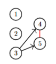
$\mathrm{forb}(m,r,M) - (r-1)^m - m(r-1)^{m-1} \le (r-2)^{m-2} H(m, \frac{r-1}{r-2})$.

Associated multigraph $\mathcal{G}_{\mathcal{B}}$

Definition

- Given a choice $\mathcal{B}$ on $[m]$ and $X \subset [m]$, the directed multigraph $\mathcal{D}_{\mathcal{B}}(X)$ associated to $\mathcal{B}$ is obtained by drawing all the arrows corresponding to 0-implications imposed by matrices $B_{i,j,k}$.
- If $\mathcal{B}$ is a "good" choice, the undirected multigraph $\mathcal{G}_{\mathcal{B}}$ associated to $\mathcal{B}$ is the "complement" of $\mathcal{D}_{\mathcal{B}}([m])$, which is a TCM.

Example

$B_{1,2,3}$	$B_{1,2,4}$	$B_{1,2,5}$	$B_{1,3,4}$	$B_{1,3,5}$	$B_{1,4,5}$	$B_{2,3,4}$	$B_{2,3,5}$	$B_{2,4,5}$	$B_{3,4,5}$
A_1	A_2	A_2	A_2	A_2	A_1	A_2	A_2	A_1	A_1
									

Relating the two problems

Theorem (Peaslee, Sali, Y., 2023+)

$\mathrm{forb}(m, r, M) - (r-1)^m - m(r-1)^{m-1} \leq (r-2)^{m-2} H(m, \frac{r-1}{r-2})$.

Proof sketch.

For every pair ij in $[m]$, let m_{ij} be the multiplicity of ij in $\mathcal{G}_{\mathcal{B}}$. Then

$$\sum_{X \subset [m]} n(\mathcal{B}, X)(r-2)^{m-|X|} \quad = \quad (r-2)^{m-2} \sum_{ij} \left(\frac{r-1}{r-2}\right)^{m_{ij}},$$

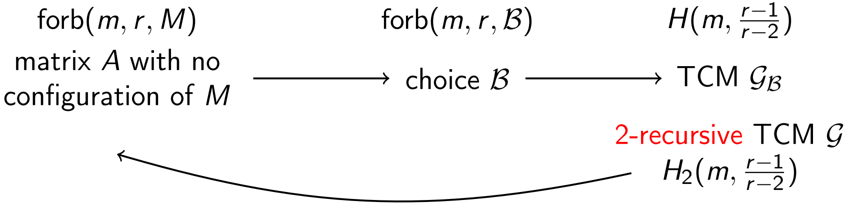
roughly because no ij edge in $\mathcal{D}_{\mathcal{B}}(X) \iff$ a copy of edge ij in $\mathcal{G}_{\mathcal{B}}$.

It then follows from the definition of $H(m, \frac{r-1}{r-2})$ because $\mathcal{G}_{\mathcal{B}}$ is a TCM. ☐

Relating the two problems

Theorem (Peaslee, Sali, Y., 2023+)

$\text{forb}(m, r, M) - (r-1)^m - m(r-1)^{m-1} \leq (r-2)^{m-2} H(m, \frac{r-1}{r-2})$.



$\text{forb}(m, r, M)$

matrix A with no configuration of M

$\text{forb}(m, r, \mathcal{B})$

choice $\mathcal{B}$

$H(m, \frac{r-1}{r-2})$

TCM $\mathcal{G}_{\mathcal{B}}$

2-recursive TCM $\mathcal{G}$

$H_2(m, \frac{r-1}{r-2})$

Theorem (Peaslee, Sali, Y., 2023+)

$\text{forb}(m, r, M) - (r-1)^m - m(r-1)^{m-1} \geq (r-2)^{m-2} H_2(m, \frac{r-1}{r-2})$.

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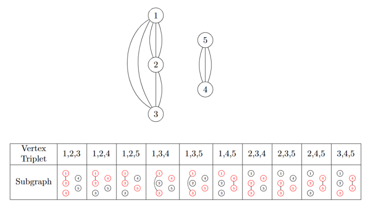
Closed sets

Definition

- A set S in a TCM $\mathcal{G}$ is closed if for every $i, j \in S$ and $k \notin S$, edge ij is chosen in triangle ijk.
- A closed set S is maximal if the only proper closed set containing S is S itself.

Example



$\{1, 2, 3\}$ and $\{4, 5\}$ are maximal closed sets, $\{1, 2\}$ is also a closed set.

Closed sets

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- A closed set S is maximal if the only proper closed set containing S is S itself.

Lemma (Peaslee, Sali, Y., 2023+)

Maximal closed sets partition the vertex set of a TCM.

Lemma (Peaslee, Sali, Y., 2023+)

If $\alpha \geq 2$, then there exists a TCM $\mathcal{G}$ maximising $w(\mathcal{G}, \alpha)$, whose maximal closed sets all have size at least 2.

Upper bound

Theorem (Peaslee, Sali, Y., 2023+)

$H(m,2) \leq \frac{83}{192} m2^{m-1} \leq 0.433 m2^{m-1}$.

Proof sketch.

- Suppose the maximal closed sets in $\mathcal{G}$ are $S_1, \cdots, S_k$, and they have sizes $2 \leq a_1 \leq \cdots \leq a_k$. Split the sum $w(\mathcal{G},2) = \sum_{ij} 2^{m_{ij}}$ according to whether pair ij is within a maximal closed set or across two of them.
- Contribution from all edges within a closed set S_ℓ is at most $H(a_\ell,2)2^{m-a_\ell}$.
- Bound the contributions from edges going across closed sets by some expression $f(a_1, \cdots, a_k)$.
- Show that $\sum_\ell H(a_\ell)2^{m-a_\ell} + f(a_1, \cdots, a_k)$ is maximised when all $a_\ell = 2$. □

Upper bound

Theorem (Peaslee, Sali, Y., 2023+)

$H(m,2) \leq \frac{83}{192} m2^{m-1} \leq 0.433 m2^{m-1}$.

Theorem (Peaslee, Sali, Y., 2023+)

For all $r \geq 3$, $\mathrm{forb}(m,r,M) \leq (r-1)^m + 1.433 m (r-1)^{m-1}$.

Proof.

The $r = 3$ case directly follows from upper bound on $H(m,2)$.
For $r \geq 4$ and any column c in an r-matrix, let the 3-support of c be the set of indices i such that $c_i \in \{0,1,2\}$. Then, the number of columns with 3-support X is at most $\mathrm{forb}(|X|,3,M)(r-3)^{m-|X|}$, so
$\mathrm{forb}(m,r,M) \leq \sum_{j=0}^{m} \binom{m}{j} (r-3)^{m-j}\, \mathrm{forb}(j,3,M)$. ☐

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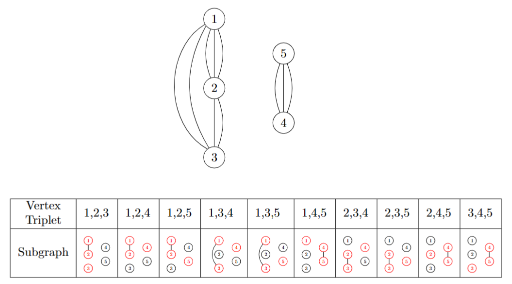
2-recursive TCM

Definition

A TCM $\mathcal{G}$ is 2-recursive if $\mathcal{G}$ has exactly two maximal closed sets, and the "restriction" of $\mathcal{G}$ to both maximal closed sets are still 2-recursive TCMs.

Example



A 2-recursive TCM on 5 vertices.

Lower bound

From the definition of 2-recursive TCM, we obtain the recurrence relation $H_2(m,\alpha) = \max\{H_2(a,\alpha)\alpha^b + H_2(b,\alpha)\alpha^b + ab\colon a+b=m\}$.

Definition

For all $\alpha > 1$, let $\lambda(\alpha) = \sum_{j=1}^{\infty} \frac{2^{j-1}}{\alpha^{2^j}}$.

Theorem (Peaslee, Sali, Y., 2023+)

- *For all* $\alpha > 1$, $\liminf_{m\to\infty} \frac{2H_2(m,\alpha)}{m\alpha^{m-1}} \geq \lambda(\alpha)$.
- *For all* $\alpha \geq 2$, $\lim_{m\to\infty} \frac{2H_2(m,\alpha)}{m\alpha^{m-1}} = \lambda(\alpha)$.

Lower bound

Theorem (Peaslee, Sali, Y., 2023+)

- *For all* $\alpha > 1$, $\liminf_{m\to\infty} \frac{2H_2(m,\alpha)}{m\alpha^{m-1}} \geq \lambda(\alpha)$.
- *For all* $\alpha \geq 2$, $\lim_{m\to\infty} \frac{2H_2(m,\alpha)}{m\alpha^{m-1}} = \lambda(\alpha)$.

Proof sketch.

For every integer m, let $k = k(m)$ be the unique integer satisfying $2^{k-1} + 2^k \leq m < 2^k + 2^{k+1}$.

- When $\alpha \geq 2$, we prove that $H_2(a,\alpha)\alpha^b + H_2(b,\alpha)\alpha^b + ab$ is maximised when $a = 2^k$ and $b = m - 2^k$.
- For other α, we obtain a lower bound by always splitting m into $2^k + (m - 2^k)$.

☐

Lower bound

Theorem (Peaslee, Sali, Y., 2023+)

For all $\alpha > 1$, $\liminf_{m\to\infty} \frac{2H_2(m,\alpha)}{m\alpha^{m-1}} \geq \lambda(\alpha)$.

Theorem (Peaslee, Sali, Y., 2023+)

For all $r \geq 3$, $\epsilon > 0$ *and all sufficiently large* m,

$$\begin{aligned}\operatorname{forb}(m, r, M) &\geq (r-1)^m + \left(1 + \frac{r-1}{2(r-2)^2}\lambda\left(\frac{r-1}{r-2}\right) - \epsilon\right) m(r-1)^{m-1} \\ &\geq (r-1)^m + \color{red}{1.360}\, m(r-1)^{m-1}\end{aligned}$$

Proof.

Follows from the theorem above and
$\operatorname{forb}(m, r, M) \geq (r-1)^m + m(r-1)^{m-1} + H_2(m, \frac{r-1}{r-2})(r-2)^{m-2}$. □

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Open questions

Conjecture (Peaslee, Sali, Y., 2023+)

For all $\alpha \geq 2$, $H(m, \alpha) = H_2(m, \alpha)$. In particular, $\text{forb}(m, 3, M) - 2^m - m2^{m-1} = H(m, 2) \sim \lambda(2)m2^{m-1}(\approx 0.391m2^{m-1})$.

- Determine the exact value, or at least the asymptotic growth of $\text{forb}(m, r, M)$ for $r \geq 3$.
- Determine the values of $H(m, \alpha)$ for every α and $H_2(m, \alpha)$ for every $\alpha < 2$.
- Determine $\text{forb}(m, r, F)$ for other (not necessarily (0,1)-) matrices F.