Counting Domino Tilings and Lozenge Tilings

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Outline

Outline

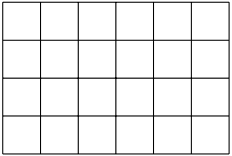
1. Domino Tilings of the Rectangle
   - Recurrence Approach
   - Adjacency Matrix Approach

2. Lozenge Tilings of the Hexagon
   - Plane Partition Approach
   - Non-Intersecting Lattice Paths Approach

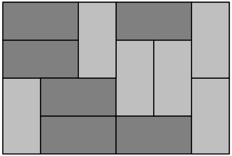
3. Further Results / Open Problems

Definition

- A domino is a 1×2 or 2×1 rectangle.
- A domino tiling of a bounded region on the square grid is a set of non-intersecting dominoes that all lie inside the region and completely covers it.



4×6 rectangle $R_{4,6}$

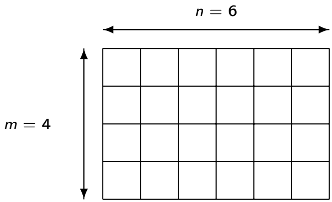


A domino tiling of $R_{4,6}$

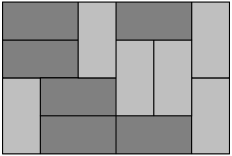
Domino Tilings

Question

Count the number of domino tilings of the $m \times n$ rectangle $R_{m,n}$.



$R_{4,6}$



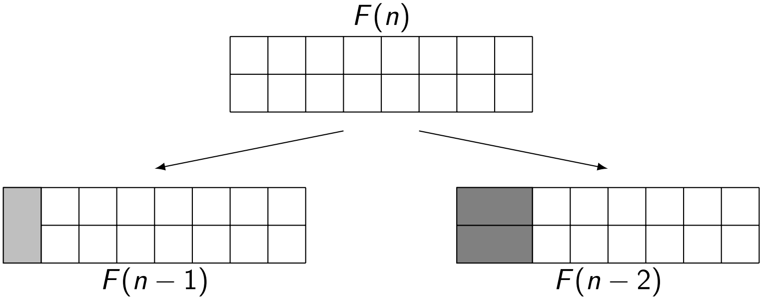
A domino tiling of $R_{4,6}$

Outline

Recurrence Approach: $2 \times n$

Question

Count the number $F(n)$ of domino tilings of $R_{2,n}$.



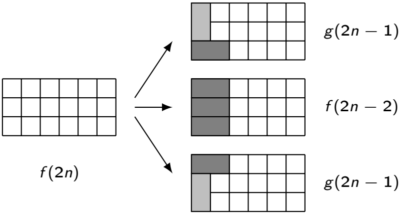
$$F(n) = F(n-1) + F(n-2)$$

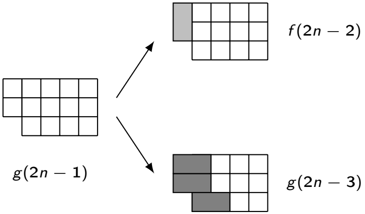
Since $F(1) = 1$ and $F(2) = 2$, $F(n)$ are the Fibonacci numbers.

Recurrence Approach: $3 \times 2n$

Question

Count the number $f(2n)$ of domino tilings of $R_{3,2n}$.





$$f(2n) = 2g(2n-1) + f(2n-2) \qquad g(2n-1) = f(2n-2) + g(2n-3)$$

Solving this, we obtain $f(2n) = 4f(2n-2) - f(2n-4)$.

Problems with the Recurrence Approach

- It quickly becomes infeasible to analyse all the possible domino positions and find the recurrence formula.
- In fact, the order of the recurrence seems to grow exponentially.

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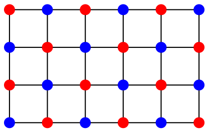
Balanced Bipartite Graph

Definition

- A graph G is bipartite if its vertices can be coloured with either red or blue, such that no edge in G connects vertices of the same colour.
- A bipartite graph is balanced if it can be coloured in this way with equal number of red and blue vertices.

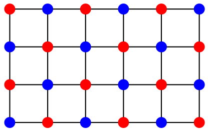


Balanced Bipartite Graph

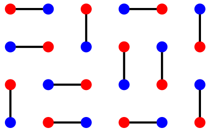
Perfect Matching

Definition

A perfect matching of a graph G is a collection M of edges in G, such that each vertex in G is contained in exactly 1 edge in M.

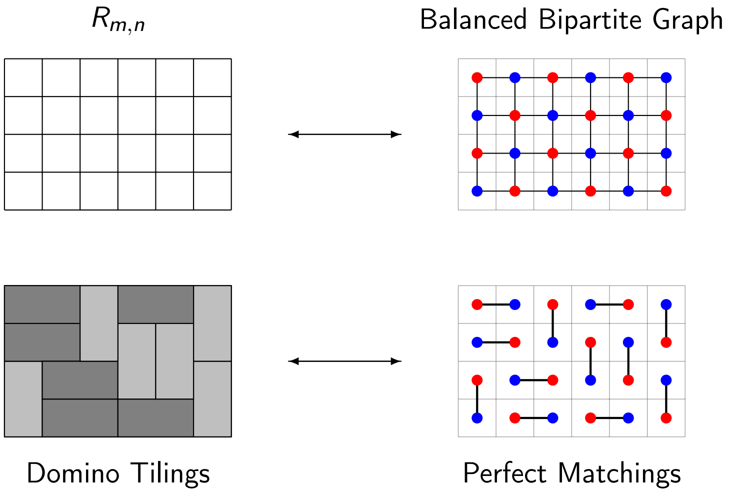


Balanced Bipartite Graph



Perfect Matching

From Domino Tilings to Perfect Matchings

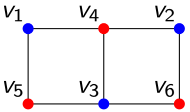


Adjacency Matrix

Definition

Let G be a balanced bipartite graph on $2n$ vertices such that $v_1, \cdots, v_n$ are coloured blue and $v_{n+1}, \cdots, v_{2n}$ are coloured red. The adjacency matrix B for G is the $n \times n$ matrix given by

$$B_{i,j} = \begin{cases} 1, & \text{if } v_i v_{n+j} \text{ is an edge in } G \\ 0, & \text{otherwise} \end{cases}$$



G

$$\longleftrightarrow \quad \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

B

Permanent and Determinant

Definition

Let M be a $n \times n$ matrix.

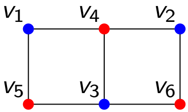
- The permanent of M is $\text{perm}(M) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} M_{i,\sigma(i)}$.
  In other words, $\text{perm}(M)$ is the sum over all possible products of n entries in M, all coming from different rows and columns.
- The determinant of M is $\det(M) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} M_{i,\sigma(i)}$.
  In other words, $\det(M)$ is the same sum as $\text{perm}(M)$, except that each term is multiplied by ± 1.

Permanent Counts Perfect Matchings

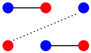
Lemma

Let G be a balanced bipartite graph and B be its adjacency matrix. Then $\mathrm{perm}(B)$ is equal to the number of perfect matchings in G.

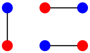
3
perfect
matchings



$$\longleftrightarrow \quad \begin{array}{c} \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \quad \mathrm{perm}(B) = 3$$



$$\begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}$$



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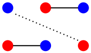

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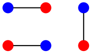
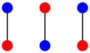


$$\begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}$$



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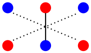


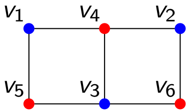
$$\begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}$$

Determinant Counts Perfect Matchings

Lemma

Let G be the graph corresponding to $R_{m,n}$ and let $\widetilde{B}$ be the matrix obtained by from the adjacency matrix B by changing the entry corresponding to every vertical edge in G from 1 to i.
Then $\left|\det(\widetilde{B})\right|$ is equal to the number of perfect matchings in G, and hence equal to the number of domino tilings of $R_{m,n}$.

3 perfect matchings



$\longleftrightarrow$

$$\begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 1 & 1 & i \\ i & 0 & 1 \\ 0 & i & 1 \end{bmatrix} \end{array} \quad \det(\widetilde{B}) = -3i$$

Kasteleyn's Formula

Theorem (Kasteleyn's Formula)

The number of domino tilings of $R_{m,n}$ is

$$\prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\left(\frac{j\pi}{m+1}\right)+4\cos^2\left(\frac{k\pi}{n+1}\right)\right)^{1/4}.$$

Proof Sketch.

If $\widetilde{A}=\begin{bmatrix}0 & \widetilde{B}^{\mathsf{T}}\\ \widetilde{B} & 0\end{bmatrix}$, then $\left|\det(\widetilde{A})\right|=\left|\det(\widetilde{B})\right|^2=(\text{number of tilings})^2$.

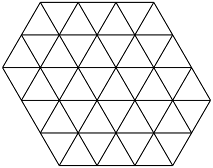
It can be shown that the mn eigenvalues of $\widetilde{A}$ are exactly $2\cos\left(\frac{j\pi}{m+1}\right)+2i\cos\left(\frac{k\pi}{n+1}\right)$ for $1\le j\le m$ and $1\le k\le n$. $\square$

Outline

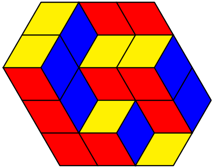
Lozenge Tilings

Definition

- A lozenge is the shape obtained by gluing two equilateral triangles along one of their sides.
- A lozenge tiling of a bounded region in the triangle grid is a set of non-intersecting lozenges that all lie inside the region and completely covers it.



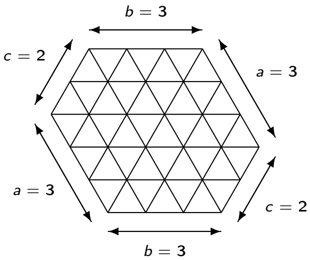
Hexagon $H_{3,3,2}$



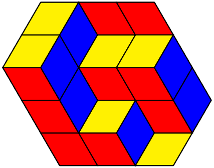
A lozenge tiling of $H_{3,3,2}$

Question

Count the number of lozenge tilings of the hexagon $H_{a,b,c}$.



$H_{3,3,2}$



A lozenge tiling of $H_{3,3,2}$

Outline

Plane Partition Approach

Definition

A plane partition $\pi = (\pi_{i,j})_{i,j=1}^{\infty}$ is a two dimensional array of non-negative integers such that

- Only finitely many $\pi_{i,j}$ are non-zero.
- $\pi_{i,j}$ is non-increasing in both indices.

$$\begin{array}{lllll} 3 & 2 & 2 & 0 & \cdots \\ 2 & 1 & 0 & \cdots & \\ 1 & 1 & 0 & \cdots & \\ 0 & 0 & \cdots & & \\ \cdots & & & & \end{array}$$

A plane partition.

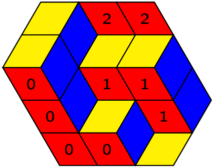
From Lozenge Tilings to Plane Partitions

Theorem

Let $\mathcal{B}(a, b, c)$ be the set of plane partitions such that

- *If $\pi_{i,j} \neq 0$, then $i \leq a$ and $j \leq b$.*
- *$\pi_{i,j} \leq c$ for all i, j.*

Then there is a bijection between lozenge tilings of $H_{a,b,c}$ and plane partitions in $\mathcal{B}(a, b, c)$.



Lozenge tiling of $H_{3,3,2}$

2	1	1
2	1	0
0	0	0

$\pi \in \mathcal{B}(3, 3, 2)$

MacMahon's Formula

Theorem (MacMahon's Formula)

It can be shown using the method of generating function that

$$|\mathcal{B}(a,b,c)| = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2} = \prod_{i=1}^{a}\prod_{j=1}^{b}\frac{i+j+c-1}{i+j-1}.$$

Hence, the number of lozenge tilings of $H_{a,b,c}$ is also equal to this number.

In particular, the number of lozenge tilings of $H_{3,3,2}$ is

$$\frac{3\times 4\times 5\times 4\times 5\times 6\times 5\times 6\times 7}{1\times 2\times 3\times 2\times 3\times 4\times 3\times 4\times 5} = 175.$$

Outline

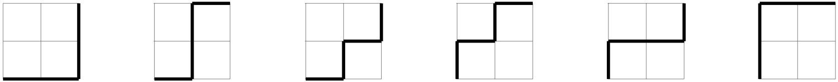
Lattice Paths

Definition

A lattice path from (a, b) to $(a + m, b + n)$ consists of $m + n$ unit-length steps, with m step going to the right and n steps going up.

Lemma

There are exactly $\binom{m+n}{m}$ lattice paths from (a, b) to $(a + m, b + n)$.

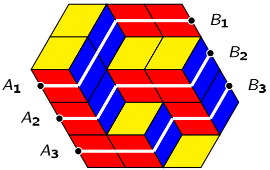


The $\binom{4}{2} = 6$ lattice paths from $(0, 0)$ to $(2, 2)$.

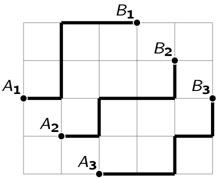
From Lozenge Tilings to Lattice Paths

Theorem

There is a bijection between lozenge tilings of $H_{a,b,c}$ and a-tuples of non-intersecting lattices paths $(P_1, \cdots, P_a)$, where P_i goes from $A_i = (i-1, a-i)$ to $B_i = (b+i-1, c+a-i)$.



$\longleftrightarrow$



Lozenge Tiling
of $H_{3,3,2}$

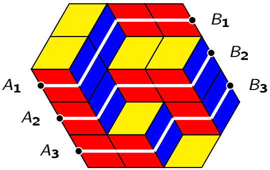
Non-Intersecting Lattice Paths

$P_1 : (0,2) \to (3,4)$
$P_2 : (1,1) \to (4,3)$
$P_3 : (2,0) \to (5,2)$

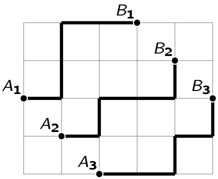
Lindström-Gessel-Viennot Lemma

Lemma (Lindström-Gessel-Viennot Lemma)

Let $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$ be lattice points in "good position". Suppose $M_{i,j}$ is the number of lattice paths from A_i to B_j and let M be the $n \times n$ matrix whose (i,j)-entry is $M_{i,j}$. Then the number of non-intersecting lattice paths $(P_1, \cdots, P_n)$ with P_i connecting A_i to B_i for each i is $\det(M)$.



$\longleftrightarrow$



$\longleftrightarrow$

$$\begin{bmatrix} \binom{5}{3} & \binom{5}{4} & \binom{5}{5} \\ \binom{5}{2} & \binom{5}{3} & \binom{5}{4} \\ \binom{5}{1} & \binom{5}{2} & \binom{5}{3} \end{bmatrix}$$

Lozenge Tiling of $H_{3,3,2}$

Non-Intersecting Lattice Paths P_1, P_2, P_3

$\det(M) = 175$

MacMahon's Formula revisited

Theorem (MacMahon's Formula)

For positive integers a, b, c, let M be the $a \times a$ matrix with $M_{i,j} = \binom{b+c}{b+j-i}$. Then the number of lozenge tilings of $H_{a,b,c}$ is equal to

$$\det(M) = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} = \prod_{i=1}^{a}\prod_{j=1}^{b} \frac{i+j+c-1}{i+j-1}.$$

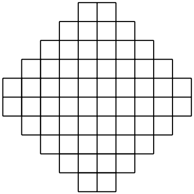
Remark

See the excellent paper Advanced Determinant Calculus by Christian Krattenthaler for a comprehensive guide on evaluating complicated determinant, especially those involving binomial coefficients or arising from tiling problems.

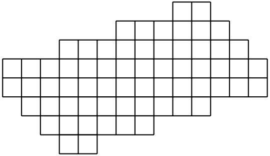
Outline

Domino

Aztec Diamond AZ_5



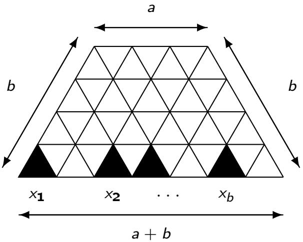
Aztec Pillow AP_4

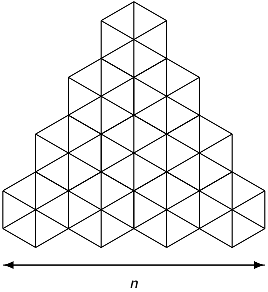


It is known that AZ_n has $2^{\frac{n(n+1)}{2}}$ domino tilings

Number of domino tilings for AP_n is only conjectured

Lozenge





Number of lozenge tilings is

$$\prod_{1 \le i < j \le b} \frac{x_j - x_i}{j - i}.$$

Explicit formula for the number of lozenge tilings is unknown.