Pattern Avoidance in Parking Functions

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Outline

Pattern avoidance in permutations

Definition

For $m \leq n$ and $\sigma \in S_m$, $\pi \in S_n$, we say that π contains σ as a pattern if there exists $1 \leq i_1 < \cdots < i_m \leq n$, such that $\pi(i_a) < \pi(i_b)$ if and only if $\sigma(a) < \sigma(b)$ for all $a, b \in [m]$, and we say π avoids σ otherwise.

Define $\mathrm{Av}_n(\sigma_1, \cdots, \sigma_k)$ to be the set of permutations in S_n avoiding all of $\sigma_1, \cdots, \sigma_k$.

Example

- The permutation 625134 contains the pattern 132, but not the pattern 1234.
- $\mathrm{Av}_n(21) = \{12 \cdots n\}$.

Parking functions

One by one, n cars enter a one-way parking lot with n parking spots.

For each $i \in [n]$, the i-th car drives straight to the $f(i)$-th parking spot, and parks there if it is still available.

Otherwise, it continues down the parking lot and parks at the first available spot, or exits without parking if there isn't one.

A function $f : [n] \to [n]$ is a parking function if all n cars park successfully.

An example parking function

i	1	2	3	4	5	6
$f(i)$	4	2	4	5	2	1

			1		

$\rightarrow$

	2		1		

$\rightarrow$

	2		1	3	

$\rightarrow$

	2		1	3	4

$\rightarrow$

	2	5	1	3	4

$\rightarrow$

6	2	5	1	3	4

Basic results on parking functions

Lemma

$f : [n] \to [n]$ is a parking function if and only if for every $i \in [n]$, at least i cars prefer one of the first i parking spots.

We could equivalently define $f : [n] \to [n]$ to be a parking function if for every $i \in [n]$,

$$|f^{-1}(\{1, 2, \cdots, i\})| \geq i.$$

Theorem (Konheim, Weiss 1966)

There are exactly $(n+1)^{n-1}$ parking functions $f : [n] \to [n]$.

There are many possible proofs:
e.g. algebraic methods, via bijections, and a proof from The Book.

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Parking permutations

Definition

For any parking function $f : [n] \to [n]$, its associated parking permutation is the permutation $\rho_f \in S_n$ satisfying that the i-th spot in the parking lot is occupied by the $\rho_f(i)$-th car.

Example

i	1	2	3	4	5	6
$f(i)$	4	2	4	5	2	1

6	2	5	1	3	4

$$\rho_f = 625134$$

Pattern avoidance in associated parking permutations

Definition

For a collection $\sigma_1, \cdots, \sigma_k$ of permutations, let $\mathrm{Pk}_n(\sigma_1, \cdots, \sigma_k)$ be the set of parking function $f : [n] \to [n]$ such that ρ_f contains none of $\sigma_1, \cdots, \sigma_k$ as a pattern. Let $\mathrm{pk}_n(\sigma_1, \cdots, \sigma_k) = |\,\mathrm{Pk}_n(\sigma_1, \cdots, \sigma_k)|$.

Example

i	1	2	3	4	5	6
$f(i)$	4	2	4	5	2	1

6	2	5	1	3	4

$$\rho_f = 625134$$

$$f \notin \mathrm{Pk}_n(132), \qquad f \in \mathrm{Pk}_n(1234).$$

A useful lemma

Lemma

For any $\rho \in S_n$ *and* $i \in [n]$, *let*

$$\ell(i,\rho) = \max\{\ell \mid \rho(j) \le \rho(i) \text{ for all } i-\ell+1 \le j \le i\},$$

$$\ell(\rho) = \prod_{i=1}^{n} \ell(i,\rho).$$

Then, $\ell(\rho)$ *is the number of parking function* $f : [n] \to [n]$ *with* $\rho_f = \rho$.

Example

$$\ell(625134) = 1 \times 1 \times 2 \times 1 \times 2 \times 3 = 12.$$

A useful lemma

Lemma

For any $\rho \in S_n$, $\ell(\rho) = \prod_{i=1}^{n} \ell(i, \rho)$ *is the number of parking function* $f : [n] \to [n]$ *with* $\rho_f = \rho$.

Corollary

$$\mathrm{pk}_n(\sigma_1, \cdots, \sigma_k) = \sum_{\rho \in \mathrm{Av}_n(\sigma_1, \cdots, \sigma_k)} \ell(\rho) = \sum_{\rho \in \mathrm{Av}_n(\sigma_1, \cdots, \sigma_k)} \prod_{i=1}^{n} \ell(i, \rho).$$

Example

$\mathrm{Av}_n(21) = \{12 \cdots n\}$, so $\mathrm{pk}_n(21) = \ell(12 \cdots n) = n!$.

Summary of results

We computed $\mathrm{pk}_n(\sigma_1, \cdots, \sigma_k)$ for all collections $\sigma_1, \cdots, \sigma_k$ of permutations of length 3, and obtained an explicit formula in every case except for $\mathrm{pk}_n(\sigma)$ with $\sigma \in \{132, 231, 312, 321\}$.

General recipe:

- Study the structure of $\mathrm{Av}_n(\sigma_1, \cdots, \sigma_k)$.
- Taking into account $\ell(\rho)$, obtain a recurrence for $\mathrm{pk}_n(\sigma_1, \cdots, \sigma_k)$.
- Solve the recurrence.

Generally, avoiding more permutations makes the problem easier.

Example: $\mathrm{pk}_n(123, 132, 213)$

Theorem (Y., 2024+)

$\mathrm{pk}_n(123, 132, 213) = \frac{1}{3}(2^{n+1} + (-1)^n)$.

Proof sketch.

- Show that $\mathrm{Av}_n(123, 132, 213)$ consists exactly of those permutations of the forms $n\rho_1$ and $n-1n\rho_2$, where $\rho_1 \in \mathrm{Av}_{n-1}(123, 132, 213)$ and $\rho_2 \in \mathrm{Av}_{n-2}(123, 132, 213)$.
- It follows that $\mathrm{pk}_n(123, 132, 213) = \mathrm{pk}_{n-1}(123, 132, 213) + 2\,\mathrm{pk}_{n-2}(123, 132, 213)$.
- Solve the linear recurrence. □

Example: $\mathrm{pk}_n(123)$

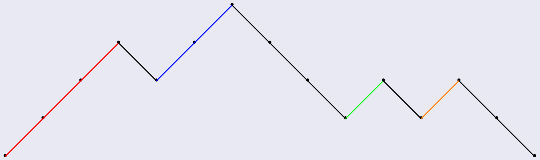
Theorem (Y., 2024+)

$\mathrm{pk}_n(123) = \frac{1}{n+1}\sum_{k=1}^{n}\binom{n+1}{k}\binom{n+k-1}{2k-1}$.

Proof sketch.

- There is a bijection mapping every $\rho \in \mathrm{Av}_n(123)$ to a Catalan path C of length $2n$, such that $\ell(\rho)$ is equal to the product of the lengths of each block of up-steps in C.

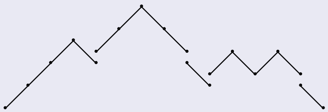
$\rho = 5471632$



Example: $\mathrm{pk}_n(123)$

Proof sketch (continued).

- There is a bijection mapping every $\rho \in \mathrm{Av}_n(123)$ to a Catalan path C of length $2n$, such that $\ell(\rho)$ is equal to the product of the lengths of each block of up-steps in C.
- Use the standard decomposition of Catalan paths to obtain a recurrence formula, and thus show that the generating function for $\mathrm{pk}_n(123)$ satisfies $P(x) = 1 + xP(x)(1 - xP(x))^{-2}$.



- Use Lagrange's Implicit Function Theorem to extract the coefficient of $P(x)$ to obtain a formula for $\mathrm{pk}_n(123)$. □