Ramsey goodness of bounded degree trees versus general graphs

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Birmingham-Warwick Combinatorics Meeting

12th December, 2023
Outline

1 Preliminaries
   - The Ramsey goodness problem
   - Known results
   - Main result

2 Base case: $k = 2$
   - $m \gg \Delta$
   - $m \ll \Delta$

3 Induction step: $k \geq 3$
   - $T$ has many leaves
   - $T$ has many bare paths and $G$ is well-connected
   - $G$ is not well-connected
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Ramsey Number

Definition (Ramsey number)

Given two graphs $H_1$ and $H_2$, the Ramsey number $R(H_1, H_2)$ is defined as the smallest integer $N$ so that for any graph $G$ with $N$ vertices, either $G$ contains a copy of $H_1$ or $G^c$ contains a copy of $H_2$.

- In general, it is difficult to give good bounds on the Ramsey number $R(H_1, H_2)$, let alone finding its exact value.
**Burr’s general lower bound**

**Definition**
For a graph $H$ with chromatic number $\chi(H)$, define $\sigma(H)$ to be the smallest possible size of a colour class in any $\chi(H)$-colouring of $H$.

**Theorem (Burr, 1981)**

Suppose $G$ is connected and $|G| \geq \sigma(H)$, then $R(G, H) \geq (|G| - 1)(\chi(H) - 1) + \sigma(H)$. 

![Graph diagram](image)
Theorem (Burr, 1981)

*Given two graphs* $G$ and $H$, *if* $G$ *is connected and* $|G| \geq \sigma(H)$, *then*

$$R(G, H) \geq (|G| - 1)(\chi(H) - 1) + \sigma(H).$$

Definition (Ramsey goodness)

Given graphs $G$ and $H$, $G$ is said to be $H$-good if

$$R(G, H) = (|G| - 1)(\chi(H) - 1) + \sigma(H).$$
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$P_n$ is $H$-good when ...

- $H = K_m$. [Erdős, 1947]
- $H = P_m$ and $n \geq m$. [Gerenscér, Gyárfás, 1967]
- $n \geq 4|H|$. [Pokrovskiy, Sudakov, 2017]
Known results: trees

A tree $T$ is $H$-good when ...

- $H = K_m$. [Chvátal, 1977]

- $\Delta(T) \leq \Delta$ and $|T|$ sufficiently large compared to $|H|$.
  [Erdős, Faudree, Rousseau, Schelp, 1985]

- Not when $T = K_{1,n}$ and $H = K_{2,2}$ or $K_{1,3}$.
  [Burr, Erdős, Faudree, Rousseau, Schelp, 1988]

- $\chi(H) = k$, $\Delta(T) \leq \Delta$ and $|T| \geq C_{\Delta,k}|H|\log^4 |H|$. [Balla, Pokrovskiy, Sudakov, 2018]

Balla, Pokrovskiy and Sudakov also conjectured that this $\log$ factor can be removed.
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We confirm the conjecture of Balla, Pokrovskiy and Sudakov.

**Theorem (Montgomery, Pavez-Signé, Y., 2023+)**

For any fixed $\Delta, k$, there exists a constant $C = C_{\Delta,k}$ such that for any graph $H$ and any tree $T$ satisfying $\chi(H) = k$, $\Delta(T) \leq \Delta$ and $|T| \geq C|H|$, $T$ is $H$-good. In other words, $R(T, H) = (|T| - 1)(k - 1) + \sigma(H)$. 
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For any fixed $\Delta, k$, there exists a constant $C = C_{\Delta, k}$ such that for any graph $H$ and any tree $T$ satisfying $\chi(H) = k$, $\Delta(T) \leq \Delta$ and $|T| \geq C|H|$, $T$ is $H$-good.

In other words, $R(T, H) = (|T| - 1)(k - 1) + \sigma(H)$.

Note that it suffices to prove this for all $H$ of the form $K_{m_1, \ldots, m_k}$. Because if $\sigma(H) = m_1 \leq \cdots \leq m_k$ are the colour class sizes of a $k$-colouring of $H$, then $G^c$ containing $K_{m_1, \ldots, m_k}$ will imply $G^c$ contains $H$. 
Theorem (Montgomery, Pavez-Signé, Y., 2023+)

For any fixed \( \Delta, k \), there exists a constant \( C = C_{\Delta, k} \) such that for any graph \( H \) and any tree \( T \) satisfying \( \chi(H) = k, \Delta(T) \leq \Delta \) and \( |T| \geq C|H| \), \( T \) is \( H \)-good.

In other words, \( R(T, H) = (|T| - 1)(k - 1) + \sigma(H) \).

Therefore, it suffices to prove the following, with \( \mu \) corresponding to \( 1/kC \).

Theorem (Montgomery, Pavez-Signé, Y., 2023+)

For any fixed \( \Delta, k \), there exists a constant \( \mu = \mu_{\Delta, k} \) such that for any \( m \leq \mu n \) and any tree \( T \) on \( n \) vertices satisfying \( \Delta(T) \leq \Delta \), \( T \) is \( K_{m, \mu n, \ldots, \mu n} \)-good.

In other words, \( R(T, K_{m, \mu n, \ldots, \mu n}) = (n - 1)(k - 1) + m \).
Proof Outline

Setting: $T$ is a tree on $n$ vertices with $\Delta(T) \leq \Delta$. $G$ is a graph on $(k-1)(n-1) + m$ vertices, and $G^c$ contains no copy of $K_{m,\mu n,\ldots,\mu n}$.

Goal: Find a copy of $T$ in $G$.

Outline: Induction on $k$.
- Base case $k = 2$:
  - $m \gg \Delta$ is large. Build a vortex structure. ← Focus of the talk.
  - $m \ll \Delta$ is small.
- Inductive step $k \geq 3$:
  - $T$ has many leaves.
  - $T$ has many bare paths and $G$ is well-connected.
  - $G$ is not well-connected.
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Expansion condition: \((m, m')\)-joined

**Definition**

A graph \(G\) is \((m, m')\)-joined if for any disjoint subsets \(U, U' \subset V(G)\) with \(|U| = m, |U'| = m'\), there exists an edge between \(U\) and \(U'\) in \(G\).

**Observation**

\[G^c \text{ contains no } K_{m,m'} \iff G \text{ is } (m, m')\text{-joined} \iff |N(U)| \geq |G| - m - m' \]

for every \(U \subset V(G)\) of size \(m\)
Key embedding lemma

- Setting: $G$ has $n + m - 1$ vertices and is $(m, \mu n)$-joined. $T$ is a tree with $n$ vertices and $\Delta(T) \leq \Delta$. We need to find a copy of $T$ in $G$.

- Main Tool: a vertex-by-vertex embedding technique of bounded degree trees into expander graphs.

**Lemma (Balla, Pokrovskiy, Sudakov, 2018)**

If $|G| \geq |T| + 13\Delta m + m'$, $G$ is $(m, m')$-joined and $\Delta(T) \leq \Delta$, then we can find a copy of $T$ in $G$.

- Main difficulty: manage the limited amount of spare vertices. Currently, $m - 1$ spare vertices, but $13\Delta m + \mu n$ needed.
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Idea: Use a vortex \(V(G) = V_0 \supset V_1 \supset \cdots \supset V_\ell\) to gradually reduce the number of spare vertices needed.
Vortex
Main difficulty: manage the limited amount of spare vertices. Currently, $m - 1$ spare vertices, but $13\Delta m + \mu n$ needed.

Pick a nested sequence of subsets $V(G) = V_0 \supset V_1 \supset \cdots \supset V_\ell$ of appropriate sizes uniformly at random. Using probablistic methods, we can guarantee the following conditions.

- For some $\lambda > 0$ and every $i \leq \ell - 1$, $G[V_i]$ is $(m, \lambda |V_i|)$-joined. $13\Delta m + \lambda |V_i|$ spare vertices needed, decreasing with $i$.

- For some $D \gg \Delta$, $G[V_\ell]$ is $(m/D, m/D)$-joined, only $13\Delta m/D + m/D < m - 1$ spare vertices needed.
Embed $T$ into the vortex
Embed $T$ into the vortex

Key conditions to maintain throughout the embedding process:

- $T_i$ covers all that remains in $V_i \setminus V_{i+1}$ (difficult!),
- The rest of $T_i$, including $v_i$, is in $V_{i+1} \setminus V_{i+2}$,
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A switching property

Observation

Since $m \ll \Delta$ is quite small, and the graph $G$ is $(m, \mu n)$-joined, $G$ is quite dense with at least $\Theta(n^2/m)$ edges.

If we embed a small portion $T_0$ of the tree $T$ randomly to $\phi(T_0)$ in $G$, this enables us to obtain a switching property satisfied by $\phi(T_0)$. 
A switching property

Suppose we are trying to embed a vertex $\ell$ whose parent in $T$ is $p$.
- either $\phi(p)$ has a neighbour in $G$ that is unused,
- or there exists $q \in T_0$ and an unused vertex $u \in G$, such $u$ can take the place of $\phi(q)$, freeing up $\phi(q)$ to be the image of $\ell$. 

$G$ before switch

$T$

$G$ after switch
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Using induction hypothesis

Setting: \( T \) is a tree on \( n \) vertices with \( \Delta(T) \leq \Delta \). \( G \) is a graph on \((k - 1)(n - 1) + m\) vertices, and \( G^c \) contains no copy of \( K_{m, \mu n, \ldots, \mu n} \).

Need to find a copy of \( T \) in \( G \).

**Lemma**

Either \( G \) contains a copy of \( T \), or \( G \) is \((m, (k - 2)(n - 1) + \mu n)-\text{joined} \).

\[ (k - 2)(n - 1) + \mu n \]

- \( G[V] \) cannot contain \( T \) as \( G \) doesn’t
- \( G[V]^c \) cannot contain \( K_{\mu n, \ldots, \mu n} \) otherwise \( G^c \) contains \( K_{m, \mu n, \ldots, \mu n} \)
  - this contradicts induction applied to \( G[V] \)
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Definition
A path $P$ in a tree $T$ is a **bare path** if all vertices in $P$ has degree exactly 2.

Lemma (Krivelevich, 2010)
Let $T$ be a tree on $n$ vertices, then
- either $T$ contains at least $\ell$ leaves,
- or $T$ contains at least $\frac{n}{s+1} - 2\ell$ bare paths of length $s$. 

Embedding $T$ with many leaves

- Remove a set $L$ of leaves, such that each $\ell \in L$ has a distinct parent in $T$ and $|L| = \Theta(n)$.
- Now $|G| \geq |T - L| + 13\Delta m + (k - 2)(n - 1) + \mu n$, so we can find an embedding $\phi$ of $T - L$.
- To add the leaves in, use expansion properties to show Hall’s matching conditions hold between $\phi(P)$ and the set $U$ of unused vertices.
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A connecting property

Definition

\( G \) is **well-connected** if for any partition \( V(G) = V_0 \cup V_1 \cup V_2 \) satisfying \( |V_0| \leq \lambda n \) and \( |V_1|, |V_2| \geq m \), there exists an edge between \( V_1 \) and \( V_2 \).

We use this to get the following connecting property.

There exists \( \delta, \ell \) such that for any disjoint \( U, U' \subset V(G) \) of size \( m \), there are \( \delta n \) disjoint paths of the same length \( \ell \) connecting them.
Embedding $T$ with many bare paths into a well-connected $G$

- Let $\mathcal{P}$ be a large collection of bare paths in $T$.
- Use Ramsey goodness of path to find a LONG path in $G$, and divide it into a collection $\mathcal{Q}$ of shorter paths.
- Use the connecting property to embed most paths in $\mathcal{P}$ via $\mathcal{Q}$.
- Use the expansion property to embed the rest of $T$. 
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Definition

A graph $G$ on $n$ vertices is not well-connected if there exists a partition $V(G) = V_0 \cup V_1 \cup V_2$, such that

1. $|V_0| \leq \lambda n$.
2. $|V_1|, |V_2| \geq m$.
3. There is no edge between $V_1$ and $V_2$. 

![Diagram showing the partition of a graph into three sets $V_0$, $V_1$, and $V_2$ with no edge between $V_1$ and $V_2$.]
Embed $T$ into a not well-connected $G$

If $G$ is not well-connected, then one of the following is true.

- $T$ in $V_0 \cup V_1$ or $V_0 \cup V_2$
- Parts of $T$ in $V_1$ and $V_2$ connected via $t \in V_0$
- $K_{m, \mu n, \ldots, \mu n}$ in $G^c$