

# Ramsey numbers of trees

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Joint work with Richard Montgomery and Matías Pavez-Signé

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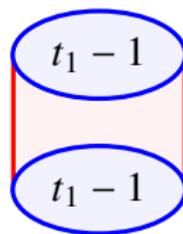
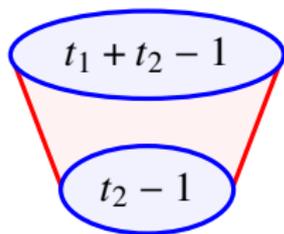
7th August, 2025

# Burr's conjecture

## Lemma

Let  $T$  be a tree with bipartition classes of sizes  $t_1 \geq t_2 \geq 2$ , then

$$R(T) \geq R_B(T) := \max\{2t_1, t_1 + 2t_2\} - 1.$$



## Conjecture (Burr, 1974)

Let  $T$  be a tree with bipartition classes of sizes  $t_1 \geq t_2 \geq 2$ , then

$$R(T) = R_B(T) = \max\{2t_1, t_1 + 2t_2\} - 1.$$

# Counterexamples

## Conjecture (Burr, 1974)

Let  $T$  be a tree with bipartition classes of sizes  $t_1 \geq t_2 \geq 2$ , then

$$R(T) = R_B(T) = \max\{2t_1, t_1 + 2t_2\} - 1.$$

## Definition

The double star  $S_{m_1, m_2}$  is the tree obtained by joining the central vertices of the two stars  $K_{1, m_1}$  and  $K_{1, m_2}$  with an edge.

## Theorem (Norin, Sun, Zhao, 2016)

$$R(S_{2m, m}) \geq (4.2 + o(1))m \geq (1.1 + o(1))R_B(S_{2m, m}).$$

## Approximate version

### Conjecture (Burr, 1974)

Let  $T$  be a tree with bipartition classes of sizes  $t_1 \geq t_2 \geq 2$ , then

$$R(T) = R_B(T) = \max\{2t_1, t_1 + 2t_2\} - 1.$$

This is false by multiplicative factors for some double star graphs, but all known counterexamples have **large maximum degrees**.

### Theorem (Haxell, Łuczak, Tingley, 2002)

For every  $\mu > 0$ , there exists  $c > 0$  such that for every large  $n$  and every  $n$ -vertex tree  $T$  with  $\Delta(T) \leq cn$ , we have

$$R(T) \leq (1 + \mu)R_B(T).$$

# Our result

## Theorem (Haxell, Łuczak, Tingley, 2002)

For every  $\mu > 0$ , there exists  $c > 0$  such that for every large  $n$  and every  $n$ -vertex tree  $T$  with  $\Delta(T) \leq cn$ , we have

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## Theorem (Montgomery, Pavez-Signé, Y., 2025+)

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$$R(T) = R_B(T) = \max\{2t_1, t_1 + 2t_2\} - 1.$$

This confirms Burr's conjecture for all trees maximum degree at most  $cn$ .

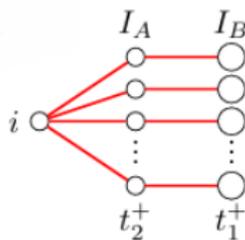
# The Haxell, Łuczak, Tingley proof sketch

## Theorem (Haxell, Łuczak, Tingley, 2002)

For every  $\mu > 0$ , there exists  $c > 0$  such that for every large  $n$  and every  $n$ -vertex tree  $T$  with  $\Delta(T) \leq cn$  and bipartition classes of sizes  $t_1 \geq t_2$ , we have

$$R(T) \leq (1 + \mu)R_B(T) = (1 + \mu) \max\{2t_1, t_1 + 2t_2\}.$$

Step 1: Find a monochromatic “HŁT structure” in the reduced graph.

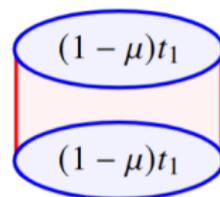
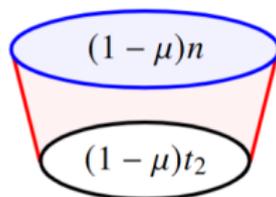


Step 2: Show that  $T$  can be embedded into the HŁT structure using **regularity**.

# Our proof sketch

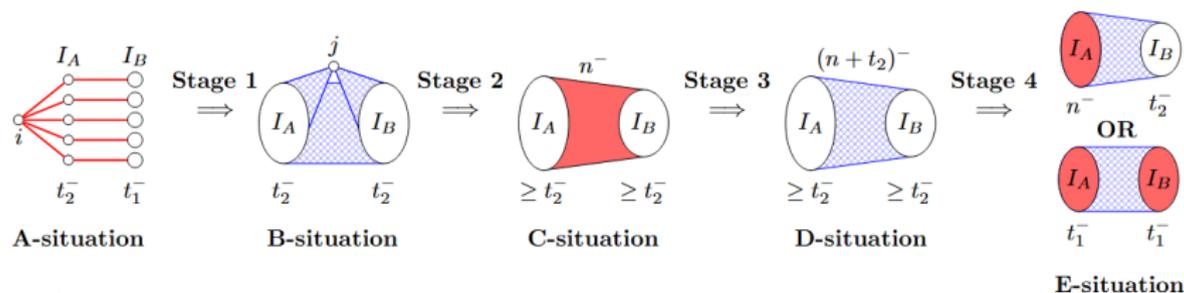
Stability part: a 4-stage process

- either we can find a structure in the reduced graph to embed a monochromatic  $T$  using regularity,
- or the reduced graph, and thus  $G$  must be **extremal**.



Extremal part: embed a monochromatic  $T$  into an **extremal** graph  $G$ .

Stability part: a 4-stage process

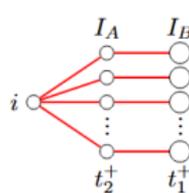


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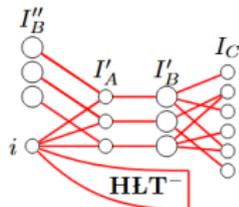
# Stability part

Stability part: a 4-stage process

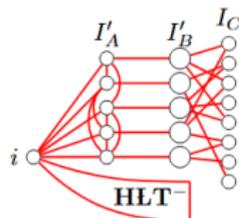
- either we can find one of the following structures in the reduced graph to embed a monochromatic  $T$  using regularity,
- or the reduced graph, and thus  $G$  must be **extremal**.



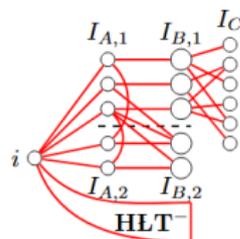
HLT



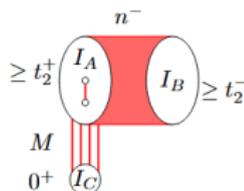
EM1a



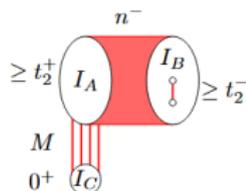
EM1b



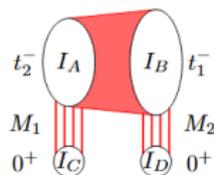
EM1c



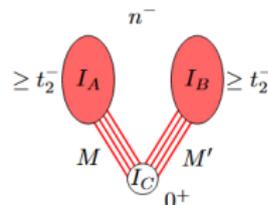
EM2a



EM2b



EM2c

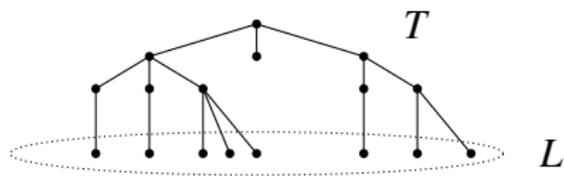
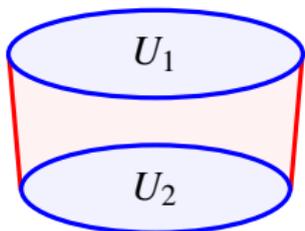


EM2d

# Extremal part: one specific case

Setting:

- $|U_1| = t_1$ ,  $|U_2| = t_1^-$ .
- $G[U_1]$ ,  $G[U_2]$  are  $\mu n$ -almost complete in blue, and  $G[U_1, U_2]$  is  $\mu n$ -almost complete in red.
- $T$  is a tree with  $n$  vertices and bipartition class sizes  $t_1 \geq t_2$ .
- $T$  contains a set  $L$  of  $\lambda n \gg \mu n$  leaves in the  $t_1$  side.

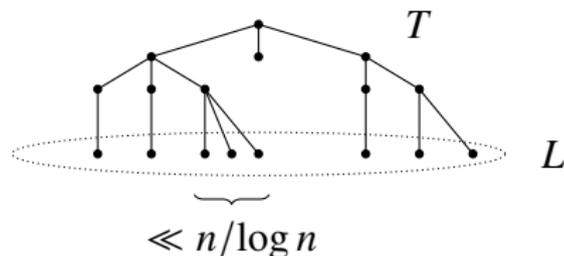
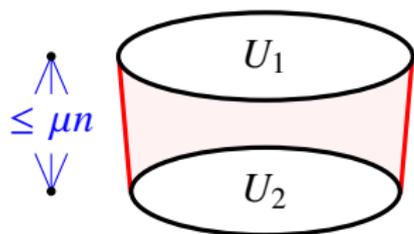


# Random embedding method: Version 1

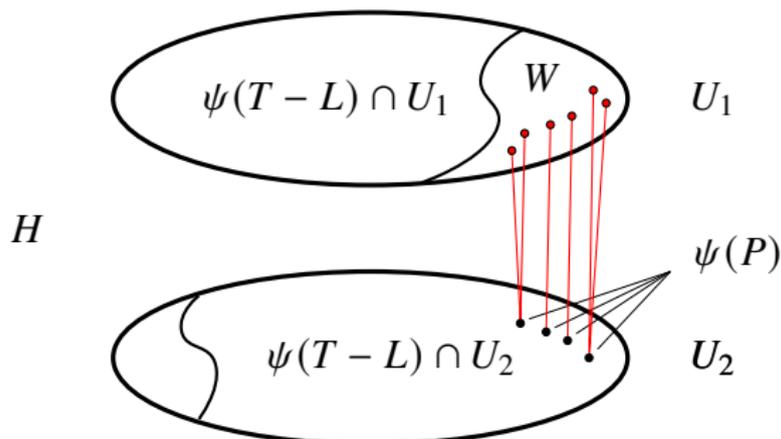
## Lemma (M, PS, Y, 2025+)

- $H = U_1 \cup U_2$  is a  $\mu n$ -almost-complete bipartite graph with  $|U_1| = t_1$ ,  $|U_2| \geq t_2 + 10\mu n$ .
- $T$  is a tree with  $n$  vertices and bipartition class sizes  $t_1 \geq t_2$ .
- $T$  contains a set  $L$  of  $\lambda n \gg \mu n$  leaves in the  $t_1$  side, such that every parent of leaves in  $L$  in  $T$  has  $d \ll n/\log n$  children in  $L$ .

Then  $H$  contains a copy of  $T$ .



# Random embedding method: Version 1 sketch



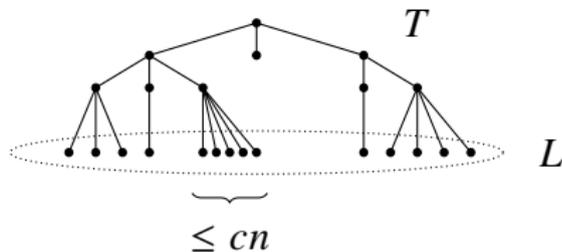
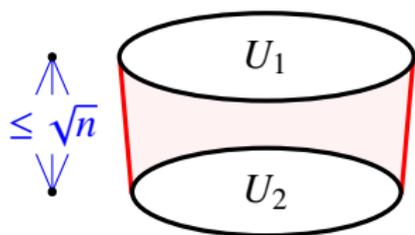
- Embed  $T - L$  between  $U_1$  and  $U_2$  **randomly**.
- Verify generalised Hall's condition to embed  $L$  into  $W$ .
  - An unused vertex  $w \in W$  is **bad** if it is adjacent to “**few**” vertices in  $\psi(P)$ .
  - Number of bad vertices is  $n \exp(-\Theta(n/d)) \ll 1$  when  $d \ll n/\log n$ .

# Random embedding method: Version 2

## Lemma (M, PS, Y, 2025+, Version 2)

- $H = U_1 \cup U_2$  is a  $\sqrt{n}$ -almost-complete bipartite graph with  $|U_1| = t_1$ ,  $|U_2| = t_2 + 10\mu n$ .
- $T$  is a tree with  $n$  vertices and bipartition class sizes  $t_1 \geq t_2$ .
- $T$  contains a set  $L$  of  $\lambda n \gg \mu n$  leaves in the  $t_1$  side, such that every parent of leaves in  $L$  has at most  $cn$  children in  $L$ .

Then  $H$  contains a copy of  $T$ .



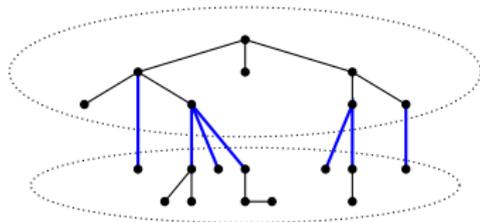
## Lemma (M, PS, Y, 2025+)

For every tree  $T$  on  $n$  vertices, there exists a partition  $V(T) = A \cup B$ , such that

- $|A|, |B| \leq (2/3 - \varepsilon)n$ ,
- $T[A]$  is a tree,
- $d(a, B) \leq \sqrt{n}$  for every  $a \in A$ ,
- some other technical conditions...

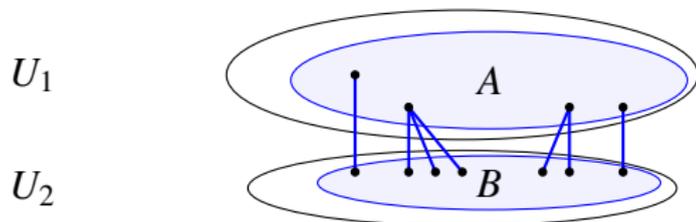
$A$

$B$



# Embedding with $\sqrt{n}$ -sparse cut

- Either  $T[A, B]$  can be embedded in blue between  $U_1$  and  $U_2$ .



- Or there are “large” subsets  $U'_1 \subset U_1$  and  $U'_2 \subset U_2$  such that  $G[U'_1, U'_2]$  is  $\sqrt{n}$ -almost complete in red.

