Ramsey numbers of trees

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Ramsey numbers of trees

Definition

The Ramsey number of a graph G, denoted as $R(G)$, is the smallest integer n such that any red/blue colouring of K_n contains a monochromatic copy of G.

Theorem

- [Gerencsér, Gyárfás, 1967] If P_{n-1} is the n-vertex path,
$$R(P_{n-1}) = \begin{cases} \frac{3n}{2} - 1 & \text{if } n \text{ is even,} \\ \frac{3n-1}{2} - 1 & \text{if } n \text{ is odd.} \end{cases}$$
- [Harary, 1972] For the n-vertex star $K_{1,n-1}$,
$$R(K_{1,n-1}) = \begin{cases} 2n-3 & \text{if } n \text{ is odd,} \\ 2n-2 & \text{if } n \text{ is even.} \end{cases}$$

Burr and Erdős' Conjecture

Conjecture (Burr, Erdős, 1976)

Let T be a tree on n vertices, then

$$R(T) \leq \begin{cases} 2n-3 & \text{if } n \text{ is odd,} \\ 2n-2 & \text{if } n \text{ is even.} \end{cases}$$

Theorem (Zhao, 2016)

Let T be a tree on n vertices with n large, then $R(T) \leq 2n-2$.

This confirms the conjecture for large even n, but the odd case is still open.

Burr's conjecture

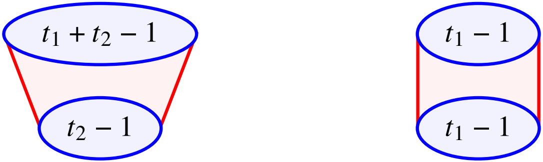
Definition

For integers $t_1 \geq t_2 \geq 1$, a (t_1, t_2)-tree is a tree whose two bipartition classes have sizes t_1 and t_2, respectively.

Lemma

Let T be a (t_1, t_2)-tree with $t_1 \geq t_2 \geq 2$, then

$$R(T) \geq \underline{R}(T) := \max\{2t_1, t_1 + 2t_2\} - 1.$$

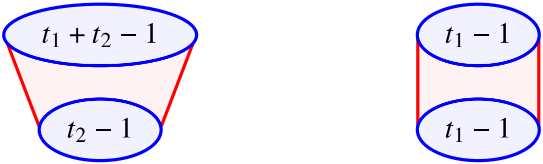


Burr's conjecture

Lemma

Let T be a tree with bipartition classes of sizes $t_1 \geq t_2 \geq 2$, then

$$R(T) \geq \underline{R}(T) := \max\{2t_1, t_1 + 2t_2\} - 1.$$



Conjecture (Burr, 1974)

Let T be a (t_1, t_2)-tree with $t_1 \geq t_2 \geq 2$, then

$$R(T) = \underline{R}(T) = \max\{2t_1, t_1 + 2t_2\} - 1.$$

Counterexamples

Conjecture (Burr, 1974)

Let T be a (t_1, t_2)-tree with $t_1 \geq t_2 \geq 2$, then

$$R(T) = \underline{R}(T) = \max\{2t_1, t_1 + 2t_2\} - 1.$$

Definition

The double star S_{m_1,m_2} is the tree obtained by joining the central vertices of the two stars K_{1,m_1} and K_{1,m_2} with an edge.

- [Grossman, Harary, Klawe, 1979]
  Burr's conjecture is false by an additive factor for some double stars.
  For example,

$$R(S_{3m,m}) = 6m + 2 = \underline{R}(S_{3m,m}) + 1.$$

Counterexamples

Definition

The double star S_{m_1,m_2} is the tree obtained by joining the central vertices of the two stars K_{1,m_1} and K_{1,m_2} with an edge.

- [Norin, Sun, Zhao, 2016]
  Burr's conjecture is false by a multiplicative factor for S_{m_1,m_2} if $m_1/m_2 \in (7/4, 105/41)$. In particular,

  $$R(S_{2m,m}) \geq (4.2 + o(1))m \geq (1.1 + o(1))\underline{R}(S_{2m,m}).$$

  Flag algebra proof $R(S_{2m,m}) \leq 4.2153m$.
- [Dubó, Stein, 2025] short elementary proof of $R(S_{2m,m}) \leq 4.275m$.

Open questions: Is $R(S_{2m,m}) = 4.2m + o(m)$?
Is there a $(2m, m)$-tree T with $R(T) \geq (4.2 + o(1))m$?

Counterexamples

- [Norin, Sun, Zhao, 2016]
  Burr's conjecture is false by a multiplicative factor for S_{m_1,m_2} if $m_1/m_2 \in (7/4, 105/41)$. In particular,

$$R(S_{2m,m}) \geq (4.2 + o(1))m \geq (1.1 + o(1))\underline{R}(S_{2m,m}).$$

- [Das, Reed, Skokan 2026+]
  For any large $t_1 \geq 2t_2$, there exists a "star-like" (t_1, t_2)-tree T such that

$$R(T) \geq \underline{R}(T) + \Theta(\max\{t_2^2/t_1, \sqrt{t_2}\}).$$

All known counterexamples have large maximum degrees.

Approximate version

Conjecture (Burr, 1974)

Let T be a tree with bipartition classes of sizes $t_1 \geq t_2 \geq 2$, then

$$R(T) = \underline{R}(T) = \max\{2t_1, t_1 + 2t_2\} - 1.$$

This is false by multiplicative factors for some double star graphs, but all known counterexamples have large maximum degrees.

Theorem (Haxell, Łuczak, Tingley, 2002)

For every $\mu > 0$, there exists $c > 0$ such that for every large n and every n-vertex tree T with $\Delta(T) \leq cn$, we have

$$R(T) \leq (1 + \mu)\underline{R}(T).$$

Our result

Theorem (Haxell, Łuczak, Tingley, 2002)

For every $\mu > 0$, there exists $c > 0$ such that for every large n and every n-vertex tree T with $\Delta(T) \leq cn$, we have

$$R(T) \leq (1+\mu)\underline{R}(T).$$

Theorem (Montgomery, Pavez-Signé, Y., 2025+)

There exists a $c > 0$ such that for every n-vertex tree T with $\Delta(T) \leq cn$ and bipartition classes of sizes $t_1 \geq t_2$, we have

$$R(T) = \underline{R}(T) = \max\{2t_1, t_1 + 2t_2\} - 1.$$

This confirms Burr's conjecture for all trees maximum degree at most cn.

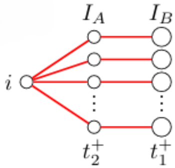
The Haxell, Łuczak, Tingley proof sketch

Theorem (Haxell, Łuczak, Tingley, 2002)

For every $\mu > 0$, there exists $c > 0$ such that for every large n and every n-vertex tree T with $\Delta(T) \leq cn$ and bipartition classes of sizes $t_1 \geq t_2$, we have

$$R(T) \leq (1+\mu)\underline{R}(T) = (1+\mu)\max\{2t_1, t_1 + 2t_2\}.$$

Step 1: Find a monochromatic "HŁT structure" in the reduced graph.

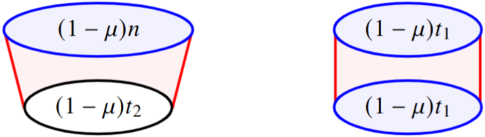


Step 2: Show that T can be embedded into the HŁT structure using regularity.

Our proof sketch

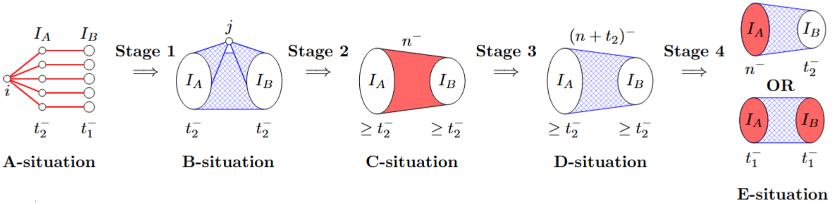
Stability part: a 4-stage process

- either we can find a nice structure in the reduced graph to embed a monochromatic T using regularity,
- or the reduced graph, and thus G must be extremal.



Extremal part: embed a monochromatic T into an extremal graph G.

Stability part: a 4-stage process

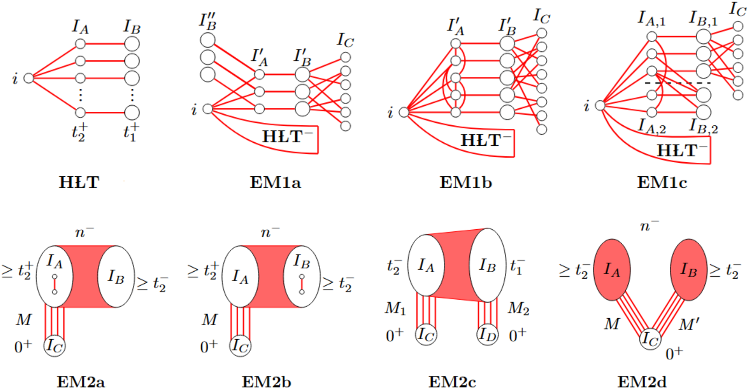


- either we can find a nice structure in the reduced graph to embed a monochromatic T using regularity,
- or the reduced graph, and thus G must be extremal.

Stability part: nice structures

Stability part: a 4-stage process

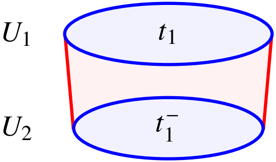
- either we can find one of the following structures in the reduced graph to embed a monochromatic T using regularity,
- or the reduced graph, and thus G must be extremal.

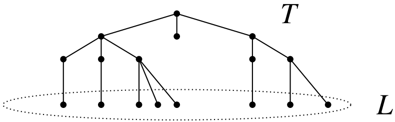


Extremal part: one specific case

Setting:

- $|U_1| = t_1$, $|U_2| = t_1^-$.
- $G[U_1]$, $G[U_2]$ are μn-almost complete in blue, and $G[U_1, U_2]$ is μn-almost complete in red.
- T is a tree with n vertices and bipartition class sizes $t_1 \geq t_2$.
- T contains a set L of $\lambda n \gg \mu n$ leaves in the t_1 side.





Challenge:

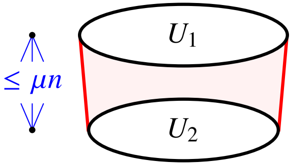
- Red: No spare vertex in U_1.
- Blue: How to go between U_1 and U_2?

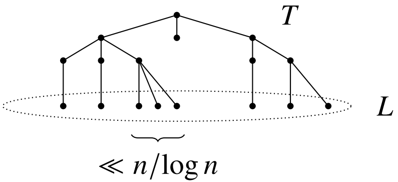
Random embedding method: Version 1

Lemma (M, PS, Y, 2025+)

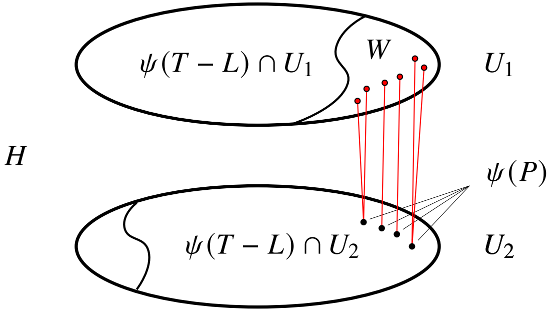
- $H = U_1 \cup U_2$ is a μn-almost-complete bipartite graph with $|U_1| = t_1$, $|U_2| \geq t_2 + 10\mu n$.
- T is a tree with n vertices and bipartition class sizes $t_1 \geq t_2$.
- T contains a set L of $\lambda n \gg \mu n$ leaves in the t_1 side, such that every parent of leaves in L in has $d \ll n/\log n$ children in L.

Then H contains a copy of T.





Random embedding method: Version 1 sketch



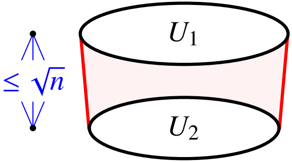
- Embed $T - L$ between U_1 and U_2 randomly.
- Verify generalised Hall's condition to embed L into W.
  - An unused vertex $w \in W$ is bad if it is adjacent to "few" vertices in $\psi(P)$.
  - Number of bad vertices is $n \exp(-\Theta(n/d)) \ll 1$ when $d \ll n/\log n$.

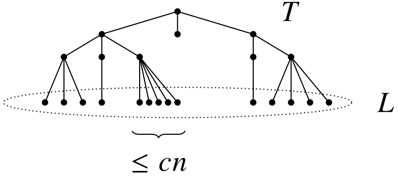
Random embedding method: Version 2

Lemma (M, PS, Y, 2025+, Version 2)

- $H = U_1 \cup U_2$ is a $\sqrt{n}$-almost-complete bipartite graph with $|U_1| = t_1$, $|U_2| = t_2 + 10\mu n$.
- T is a tree with n vertices and bipartition class sizes $t_1 \geq t_2$.
- T contains a set L of $\lambda n \gg \mu n$ leaves in the t_1 side, such that every parent of leaves in L in has at most cn children in L.

Then H contains a copy of T.



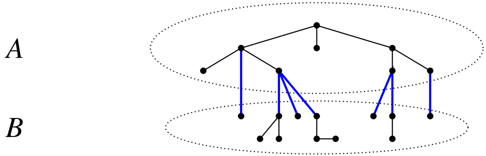


$\sqrt{n}$-sparse cut

Lemma (M, PS, Y, 2025+)

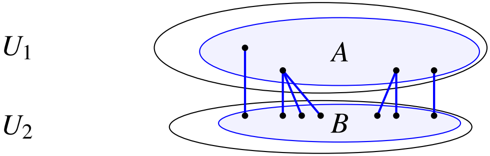
For every tree T on n vertices with $\Delta(T) \leq cn$, there exists a partition $V(T) = A \cup B$, such that

- $|A|, |B| \leq (2/3 - \varepsilon)n$,
- $T[A]$ is a tree,
- $d(a, B) \leq \sqrt{n}$ for every $a \in A$,
- some other technical conditions...

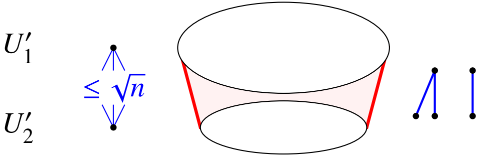


Embedding with $\sqrt{n}$-sparse cut

- Either $T[A, B]$ can be embedded in blue between U_1 and U_2.



- Or there are “large" subsets $U_1' \subset U_1$ and $U_2' \subset U_2$ such that $G[U_1', U_2']$ is $\sqrt{n}$-almost complete in red. $\Rightarrow$ Version 2.



New result

Theorem (Montgomery, Pavez-Signé, Y., 2025+)

There exists a $c > 0$ such that for every n-vertex tree T with $\Delta(T) \leq cn$ and bipartition classes of sizes $t_1 \geq t_2$, we have

$$R(T) = \underline{R}(T) = \max\{2t_1, t_1 + 2t_2\} - 1.$$

Theorem (Das, Reed, Skokan, 2026+)

Let $t_1 \geq 500t_2$, and let T be a (t_1, t_2)-tree with $\Delta(T) \leq t_1 - t_2$, then

$$R(T) = \underline{R}(T) = 2t_1 - 1.$$

Moreover, the maximum degree condition is asymptotically best possible.

- Fairly elementary proof that does not use regularity.
- Can the constant 500 be improved to 2?

Asymmetric Ramsey numbers of trees

Definition

The Ramsey number of two graphs G_1 and G_2, denoted as $R(G_1, G_2)$, is the smallest integer n such that any red/blue colouring of K_n either contains a red copy of G_1 or a blue copy of G_2.

Theorem

- [Gerencsér, Gyárfás, 1967] If $n \geq m \geq 2$,

$$R(P_{n-1}, P_{m-1}) = \begin{cases} n + \frac{m}{2} - 1 & \text{if } m \text{ is even,} \\ n + \frac{m-1}{2} - 1 & \text{if } m \text{ is odd.} \end{cases}$$

- [Harary, 1972]

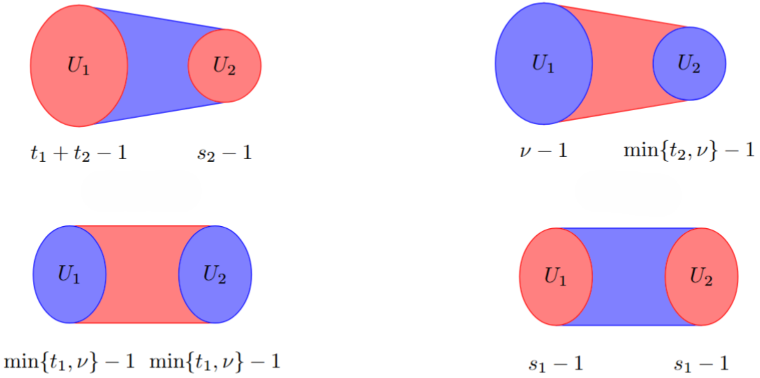
$$R(K_{1,n-1}, K_{1,m-1}) = \begin{cases} n + m - 3 & \text{if } n, m \text{ are both odd,} \\ n + m - 2 & \text{otherwise.} \end{cases}$$

Lower bounds

Lemma (Y., 2025+)

Let T be an n-vertex tree with bipartition class sizes $t_1 \geq t_2$. Let S be a ν-vertex tree with bipartition class sizes $s_1 \geq s_2$. If $n \geq \nu$, then

$$R(T,S) \geq \underline{R}(T,S) := \max\{n + s_2, \nu + \min\{t_2, \nu\}, \min\{2t_1, 2\nu\}, 2s_1\} - 1.$$



Asymmetric Ramsey numbers of trees

Theorem (Y., 2025+)

There exists a constant $c > 0$ such that the following holds.

Let $n \geq \nu$ be sufficiently large integers. Let T be an n-vertex tree with bipartition class sizes $t_1 \geq t_2$. Let S be a ν-vertex tree with bipartition class sizes $s_1 \geq s_2$.

If (i) $\Delta(T) \leq cn/\log n$ and $\Delta(S) \leq c\nu/\log \nu$, (ii) $s_2 \geq t_2$, and (iii) $\nu \geq t_1$, then

$$R(T, S) = \underline{R}(T, S) = \max\{n + s_2, 2t_1\} - 1.$$

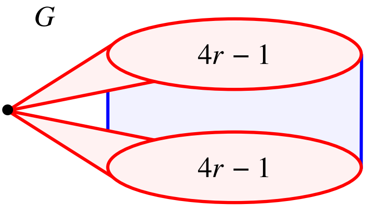
Moreover, none of the conditions (i), (ii), and (iii) can be removed.

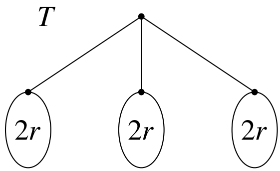
Counterexample showing (ii) is necessary

Theorem (Y., 2025+)

For any $r \gg 1$, there exist trees T and S such that the following hold.

- $\Delta(T), \Delta(S) \leq 10$.
- T has bipartition class sizes $4r$ and $2r+1$.
- S has bipartition class sizes $4r$ and $2r-1$.
- $R(T,S) \geq 8r = \underline{R}(T,S) + 1$.



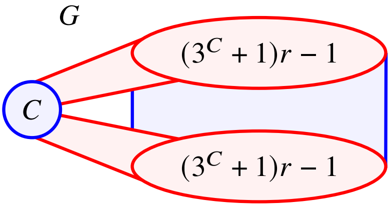


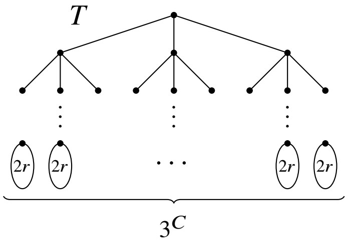
Counterexample showing (ii) is necessary

Theorem (Y., 2025+)

For any $r \gg C \geq 1$, there exist trees T and S such that the following hold.

- $\Delta(T), \Delta(S) \leq 3^{C+1}$.
- T has bipartition class sizes $(3^C + 1)r$ and $(3^C - 1)r + (3^C - 1)/2$.
- S has bipartition class sizes $(3^C + 1)r$ and $2r - (3^C - 1)/2$.
- $R(T, S) \geq 2(3^C + 1)r + C - 1 = \underline{R}(T, S) + C$.





Open questions

Theorem (Y., 2025+)

There exists a constant $c > 0$ such that the following holds.

Let $n \geq \nu$ be sufficiently large integers. Let T be an n-vertex tree with bipartition class sizes $t_1 \geq t_2$. Let S be a ν-vertex tree with bipartition class sizes $s_1 \geq s_2$.

If (i) $\Delta(T) \leq cn/\log n$ and $\Delta(S) \leq c\nu/\log \nu$, (ii) $s_2 \geq t_2$, and (iii) $\nu \geq t_1$, then

$$R(T,S) = \underline{R}(T,S) = \max\{n + s_2, 2t_1\} - 1.$$

- Can (i) be relaxed to $\Delta(T) \leq cn$ and $\Delta(S) \leq c\nu$?
- If either (ii) or (iii) is omitted, is the result above at least approximately true with suitable maximum degree conditions?