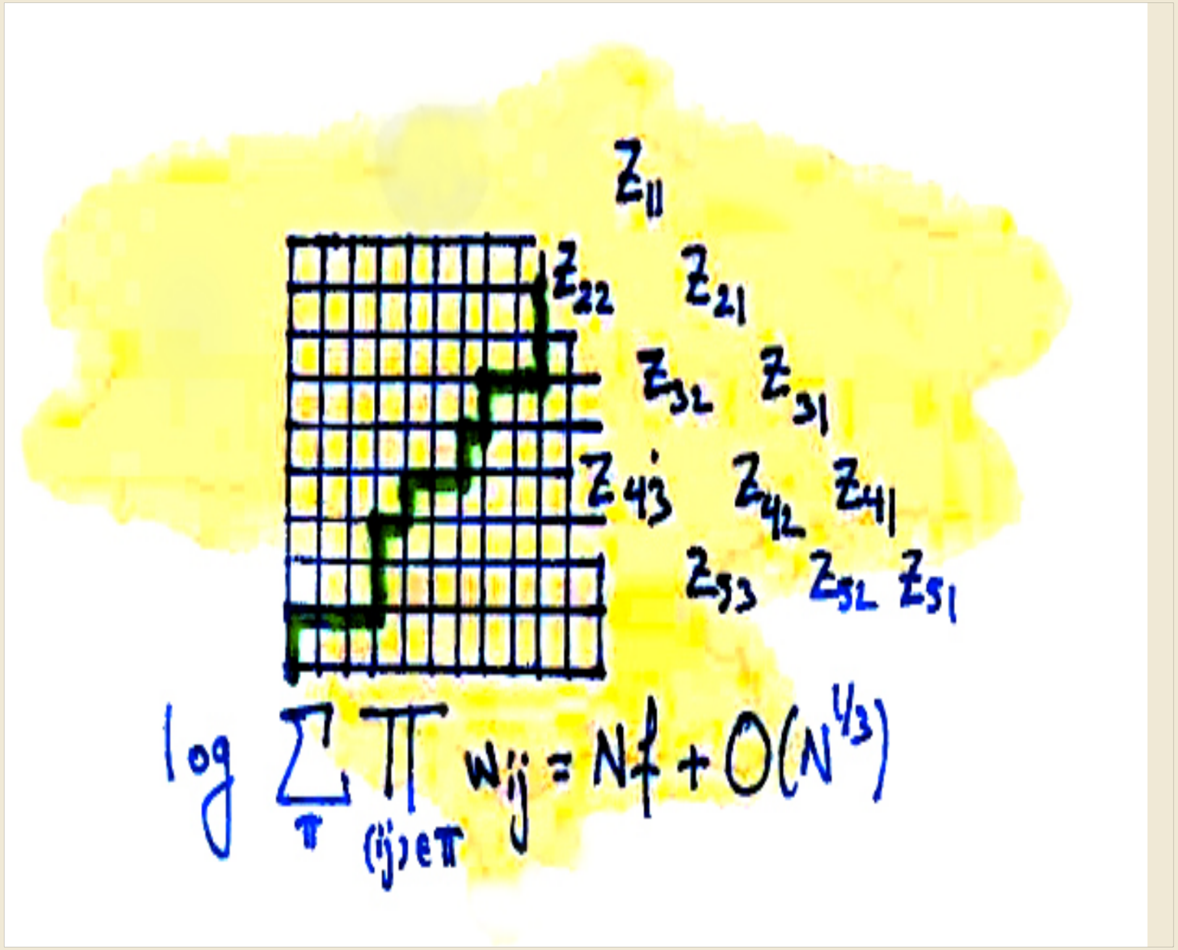
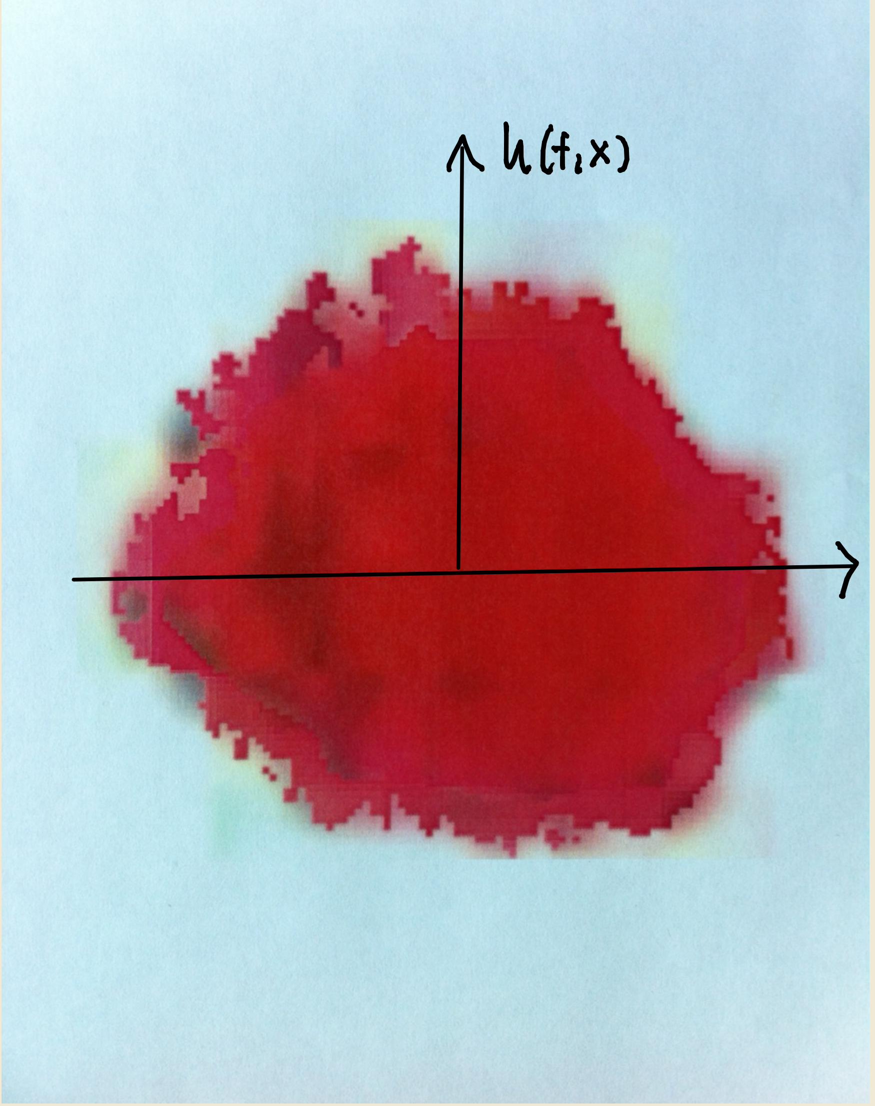


Integrable Probability

Nikos Zygouras

T.C.C.

What is the Kardar-Parisi-Zhang (KPZ) ?



$$\frac{\partial h}{\partial t} = \frac{1}{2}\Delta h + \frac{1}{2}|\nabla h|^2 + \dot{W}(t,x)$$

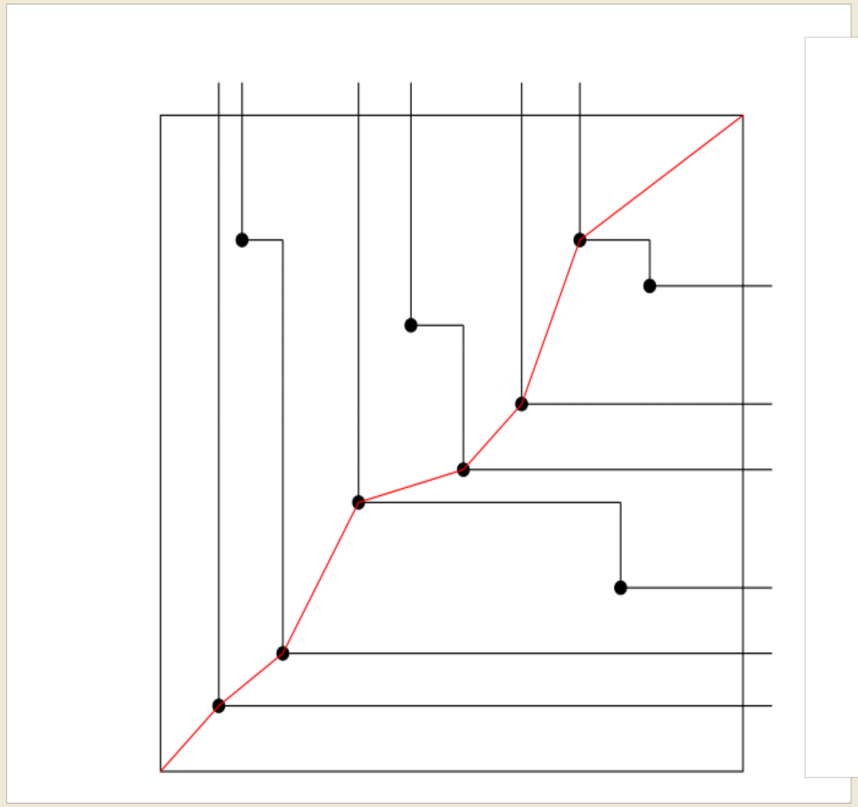
Conjecture: $h(t, t^{2/3}x) \sim t\mu + t^{1/3} \cdot \{\text{Airy process}\}$

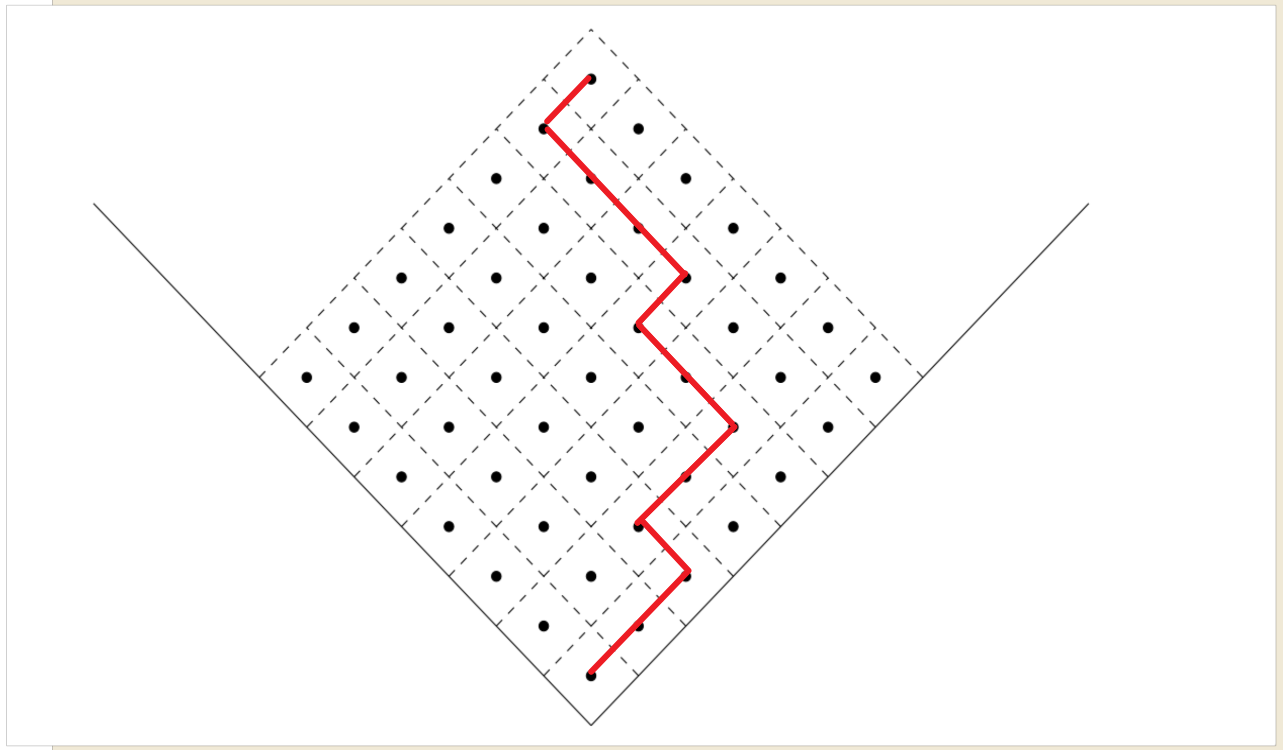
Universality Conjecture: This limit behaviour is universal among a large class of 1d systems

Higher dimensions: Even physical predictions elude us.

1d KPZ & Stochastic Integrability

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 10 & 2 & 4 & 7 & 5 & 6 & 9 & 3 & 8 \end{pmatrix}$$





Longest Increasing Subsequence

Hammersley Process

Last Passage Percolation

$$T_N := \max_{\pi} \sum_{(ij)\in\pi} w_{ij}$$

First Breakthroughs

Baik-Deift-Johansson, JAMS '99

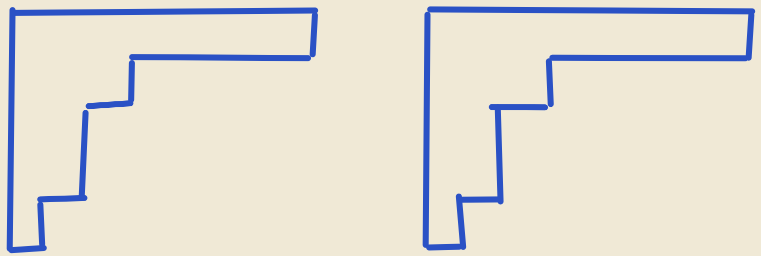
$$\frac{L_N - 2\sqrt{N}}{N^{1/6}} \xrightarrow{(d)} \text{T.W.}_{\text{GUE}}$$

Okounkov, Selecta '01 : Schur Measure

Prähofer-Spohn, Phys. Rev. Lett '00 : Airy Process

Key Steps

Robinson-Schensted-Knuth: $\left(w_{ij} \right)_{n \times m} \longmapsto$



(P) (Q)

Length of first row: $\lambda_1 = \max_{\pi} \sum_{(i,j) \in \pi} w_{ij}$

Law of (P, Q) is tractable $\mathbb{P}\left(sh(P) = \lambda \right) \propto s_\lambda(x) s_\lambda(y)$

if $w_{ij} \sim \mathrm{Exp}(x_i + y_j)$ (Integrable Probability)

Schur functions & determinantal processes

$$s_\lambda(x) = \frac{\det\left(x_i^{\lambda_j + N - j} \right)_{i,j \le N}}{\det\left(x_i^{N-j} \right)_{i,j \le N}}$$

$$\mathbb{P}\left(\tau_N \le a \right) \propto \sum_{\lambda : \lambda_1 \le a} \underbrace{s_\lambda(x)\, s_\lambda(y)}_{\text{determinantal measure}}$$

$$= \det\left(I + K_N^{LPP} \right)_{L^2(a,\infty)}$$

$$K_N^{LPP}(s,t) = \frac{1}{(2\pi i)^2} \int_{\gamma_1} d\zeta \int_{\gamma_2} d\eta \, \frac{\eta^s \zeta^t}{1 - \eta\zeta} \prod_{j=1}^{N} \frac{1 - \eta q_j}{\eta - p_j} \cdot \prod_{i=1}^{N} \frac{1 - \zeta p_i}{\zeta - p_i}$$

Beyond determinantal processes
(2008 – today)

Tracy-Widom '08-'10 [solved ASEP via Bethe Ansatz]

T.W. asymptotics for KPZ [Calabrese – Le Doussal, Dotsenko '10
Sasamoto – Spohn, Amir – Corwin – Quastel]

Quantum Toda Hamiltonian [O'Connell '10]

Log-gamma polymer [Seppäläinen '10]

Links to "Tropical Combinatorics" & Kirillov's RSK [Corwin – O'Connell – Seppäläinen – Zygouras '14
O'Connell – Seppäläinen – Zygouras '14
Nguyen – Zygouras '17]

Macdonald Processes [Borodin – Corwin '14]

Links to 6-vertex model & higher integrable systems [Borodin, Corwin, Sasamoto, Gorin, Petrov, ...]

More general models & symmetric functions

Macdonald polynomials: Eigenfunctions of Macdonald operators

$$D_n^r := \sum_{I \subset \{1,\dots,n\}} A_I(x;t) \prod_{i \in I} T_{q,x_i}$$

$$A_I(x;t) := t^{\binom{r}{2}} \prod_{i \in I,\, j \notin I} \frac{t x_i - x_j}{x_i - x_j}$$

$$T_{q,x_i} f(x_1,\dots,x_n) := f(x_1,\dots,q x_i,\dots,x_n)$$

Whittaker functions

Eigenfunctions of Quantum Toda Hamiltonian

$$\Delta - 2 \sum_{\alpha \in S} e^{-\langle \alpha, x \rangle}$$

S: positive roots of a group G

GL_n : $$\Delta - 2 \sum_{i=1}^{n} e^{-x_i + x_{i+1}}$$

SO_{2n+1} $$\Delta - 2 \sum_{i=1}^{n} e^{-x_i + x_{i+1}} - 2e^{-x_n}$$

Number theory : Fourier coefficients of automorphic forms

Mirror symmetry : sort of "Laplace transform"

Cauchy Identities

Schur $$\sum_{\lambda} s_\lambda(x)\, s_\lambda(y) = \prod_{i,j} (1 - x_i y_j)$$

$\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots\}$ partition

Macdonald $$\sum_{\lambda} P_\lambda(x)\, Q_\lambda(y) = \prod_{i,j} \frac{(t x_i y_j ; q)_\infty}{(x_i y_j ; q)_\infty}$$

$$(x;q)_\infty = \prod_{i=1}^{\infty} (1 - x q^{i-1})$$ Pochhammer symbol

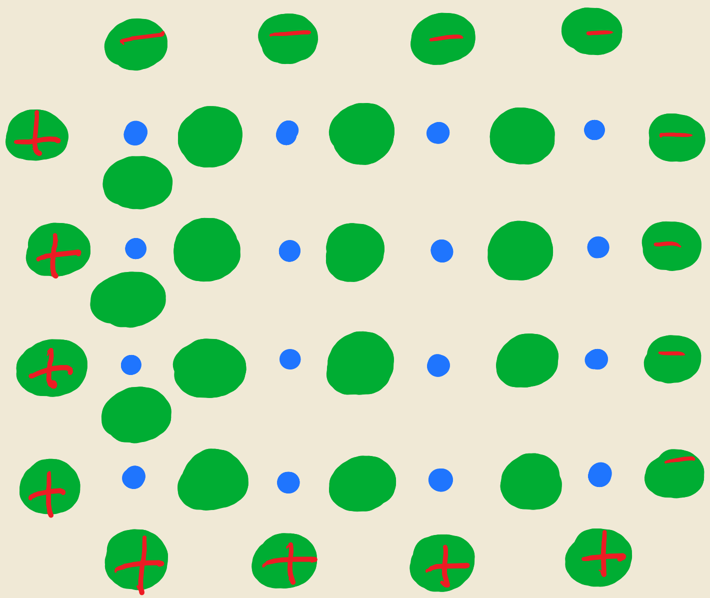
Whittaker $$\int \varphi_\alpha^{gl_n}(x)\, \varphi_\beta^{gl_n}(x)\, \frac{dx}{x} = \prod_{i,j} \Gamma(\alpha_i + \beta_j)$$

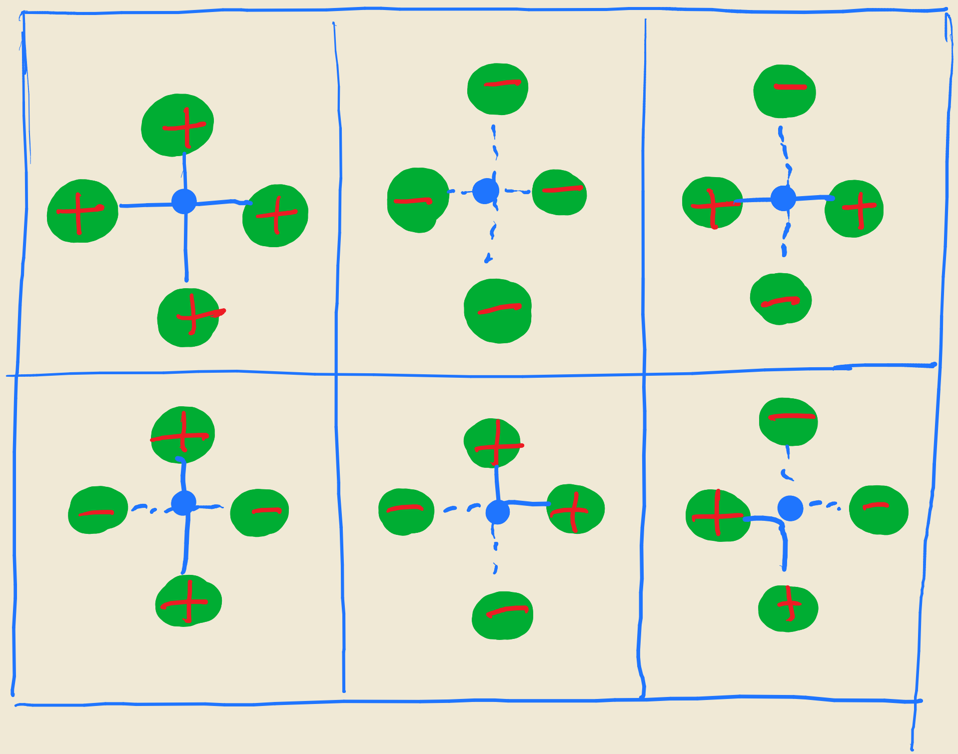
$x = (x_1, x_2, \ldots, x_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$

$\Gamma(\cdot)$ = gamma function

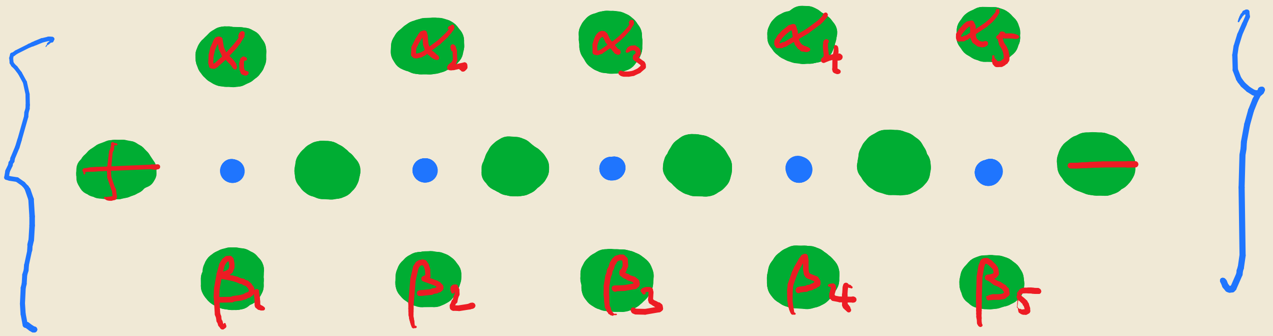
Yang-Baxter equation & vertex models

6-vertex model





Z

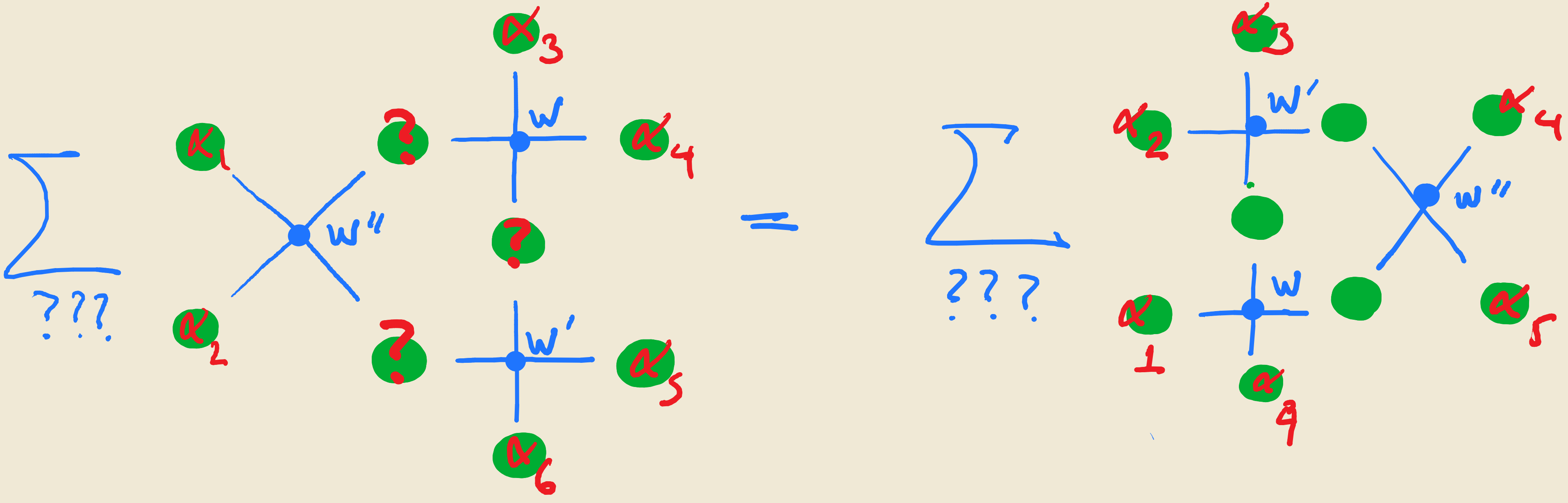


$=: V(\alpha, \beta)$

Goal: to compute multi-row partition function

$$\sum_{\alpha_1, \dots, \alpha_n} V_1(\alpha_1, \alpha_2)\, V_2(\alpha_2, \alpha_3) \cdots V_n(\alpha_n, \alpha_{n+1})$$

Yang-Baxter eq.



swap $\alpha_1 - \alpha_2$ & $w - w'$

Pieri rule &
dynamics

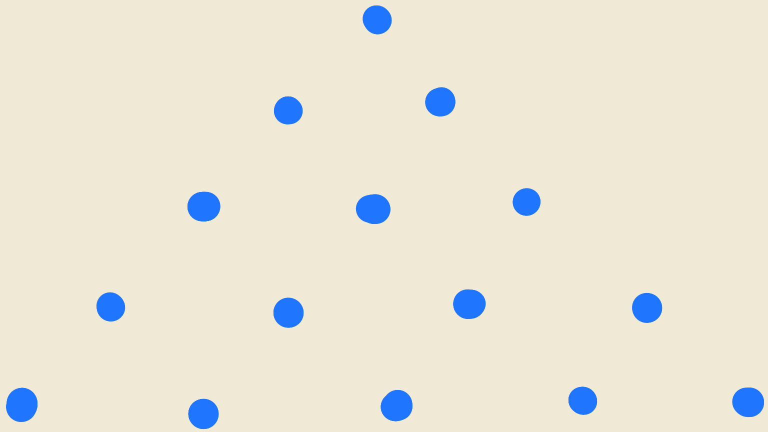
$$h_k(x_1,\dots,x_n) = \sum_{1\le i_1 \le \dots \le i_k \le n} x_{i_1}\cdots x_{i_k}$$

$$h_k s_\lambda = \sum_{\nu >_k \lambda} s_\nu$$

for $k=1$

$$\left(\sum_{i=1}^{k} x_i\right)\cdot s_\lambda = \sum_{i=1}^{n} s_{\lambda+e_i}$$

Probabilistic dynamics



$$\mathrm{Rate}\left[\begin{array}{cc} {}^{x}\bullet \curvearrowright & {}^{y}\bullet \\ {}^{\ell}\bullet & {}^{r}\bullet \curvearrowright \end{array}\right] = q^{r-x}\,\frac{1-q^{y-r}}{1-q^{y-x}}$$

$$\mathrm{Rate}\left[\begin{array}{cc} {}^{x}\bullet \curvearrowright & {}^{y}\bullet \\ {}^{\ell}\bullet \curvearrowright & \bullet^{r} \end{array}\right] = 1 - \mathrm{Rate}\left[\begin{array}{cc} {}^{x}\bullet \curvearrowright & {}^{y}\bullet \\ {}^{\ell}\bullet & {}^{r}\bullet \curvearrowright \end{array}\right]$$

Course Outline

Robinson-Schensted-Knuth

combinatorial

piece-wise & geometric

path constructions

Solvable Last Passage Percolation

Schur functions

Determinantal calculus

Asymptotics

Solvable Polymer model

Whittaker functions

Cauchy & Pieri identities

Macdonald polynomials

Markovian dynamics - particle systems

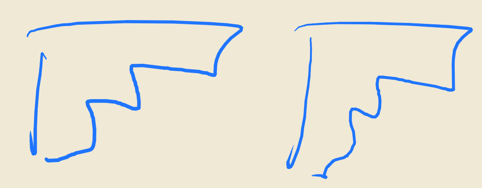
Vertex models & Yang-Baxter equation

Lecture 2: Robinson-Schensted-Knuth bijection (RSK)

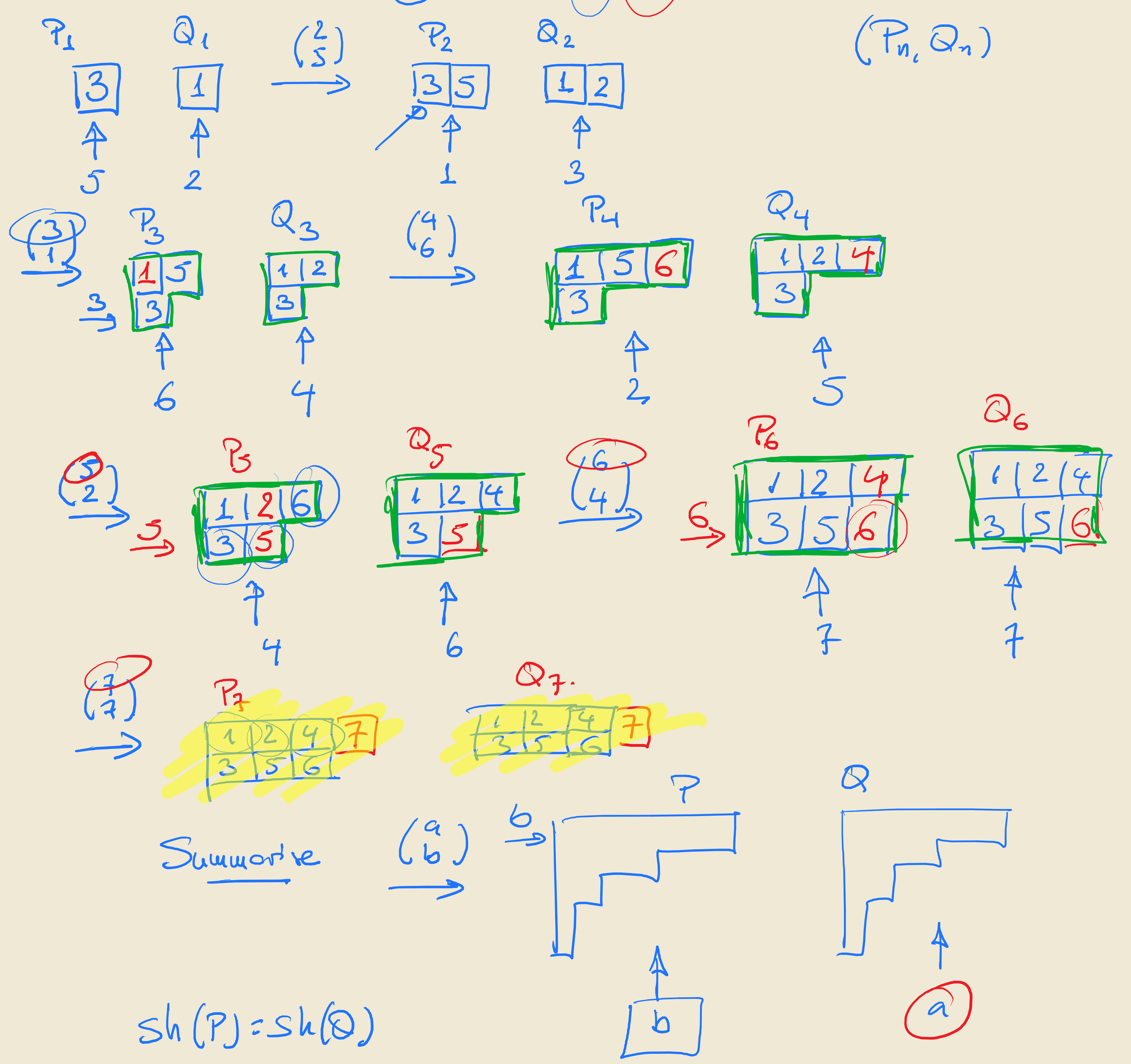
The algorithm via an example:

* RS is a bijection between permutations & Young tableaux
  RSK is —"— matrices & —"—

Example: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 1 & 6 & 2 & 4 & 7 \end{pmatrix} \longrightarrow$



(P_n, Q_n)



$sh(P) = sh(Q)$

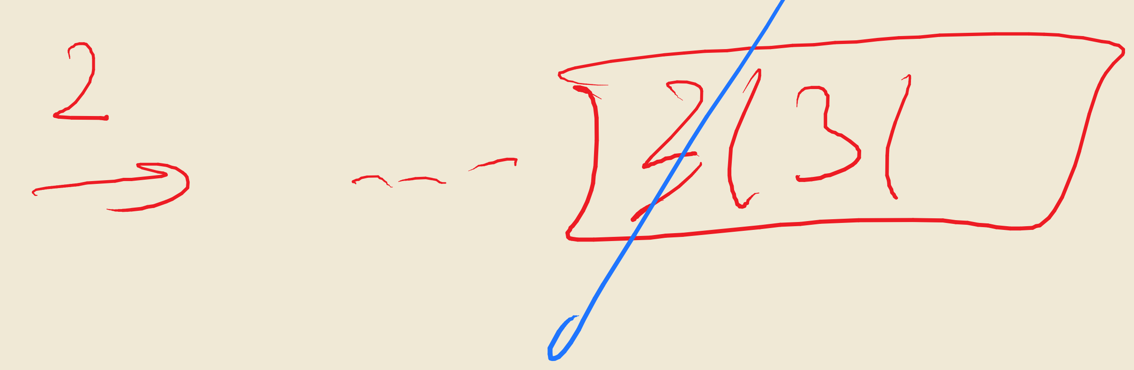
(b). finds the first number in the first row $> b$

- Kicks it out & takes its place
- the kicked out number is bumped down & tries to be inserted into the next row following the same rules
- if b is bigger than all entries in the first row then it places itself at the end of the row.

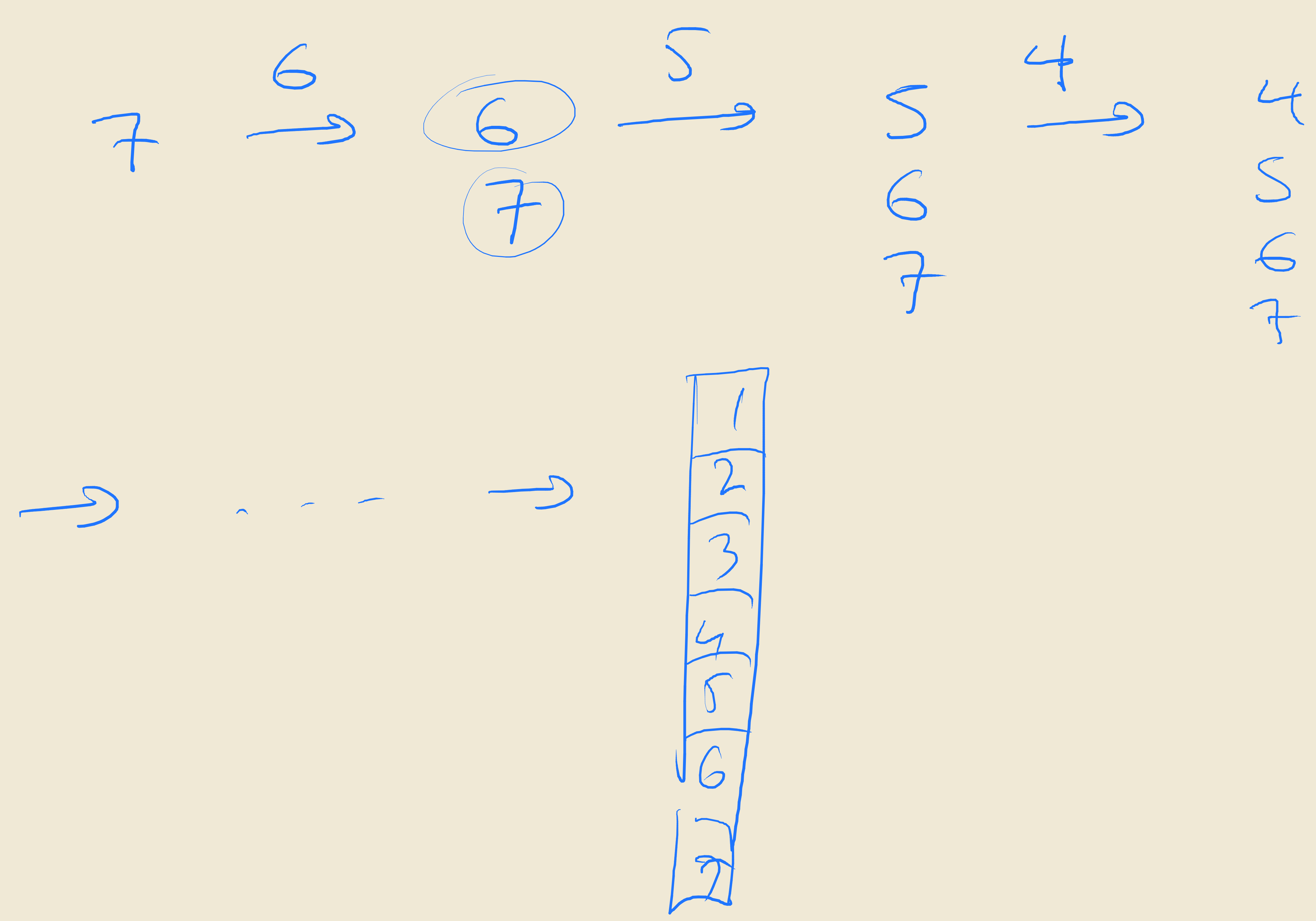
← 3 5 1 6 2 4 7

Question ?

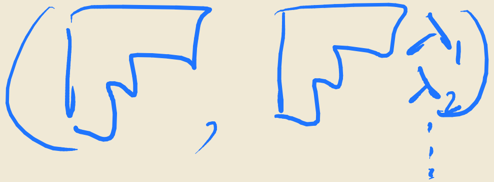
$$\begin{matrix} 1 & 1 & 2 & 2 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 1 & 2 & 2 & 5 \end{matrix}$$



$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{matrix}$$



Some properties:

$\sigma \longleftrightarrow$  $= (P, Q)$

$\lambda = (\lambda_1, \lambda_2, \dots) = sh(P) = sh(Q)$

partition

1) Let $\lambda_1 \geq \lambda_2 \geq \cdots$ be the (common) shape of YT's

then (a) λ_1 = length of longest increasing subsequence

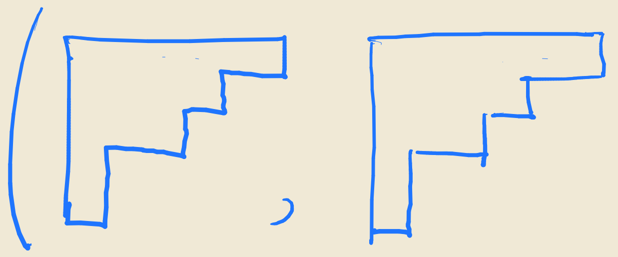
(b) $\lambda_1 + \cdots + \lambda_k$ = max length of the union of k disjoint increasing subsequences

Schensted.

2) Symmetry: if $\sigma = \sigma^{-1}$ then $P = Q$.

Piecewise linear formulation of RSK

$\to w^i_j \in \mathbb{N}$

$(w^i_j)_{\substack{i \le n \\ j \le N}} \longmapsto \Big($  $\Big) = (P, Q)$

$\underset{\text{word}}{w^i} = (\underbrace{w^i_1}, \dots, \underbrace{w^i_N}) \cong \underbrace{1 \cdots 1}_{w^i_1} \; \underbrace{2 \cdots 2}_{w^i_2} \cdots \underbrace{N \cdots N}_{w^i_N}$

Row insertion:

$x = 1^{x_1} 2^{x_2} \cdots N^{x_N}$ } input

$a = 1^{a_1} 2^{a_2} \cdots N^{a_N}$

$x \xrightarrow{a} \tilde{x}$, $\downarrow b$

$w^1 = \underbrace{1 \cdots 1}_{w^1_1} \; \underbrace{2 \cdots 2}_{w^1_2} \cdots \underbrace{N \cdots N}_{w^1_N}$

$\{\emptyset \leftarrow w^1\} = \underbrace{|1|1 \cdots 1|}_{w^1_1} \; \underbrace{2 \cdots 2|}_{w^1_2} \cdots \underbrace{|N \cdots N|}_{w^1_N}$ P_1

$P_1 \xleftarrow{RS} w^2 := \underbrace{1 \cdots 1}_{w^2_1} \; \underbrace{2 \cdots 2}_{w^2_2} \cdots \underbrace{N \cdots N}_{w^2_N}$

P_2 $|-1-|1\cdots1|2 - |\cancel{3}3 \cdot |-N-| \leftarrow w^3$

$2 \to |2|2 \cdots 2|3 - 3|$

P_3

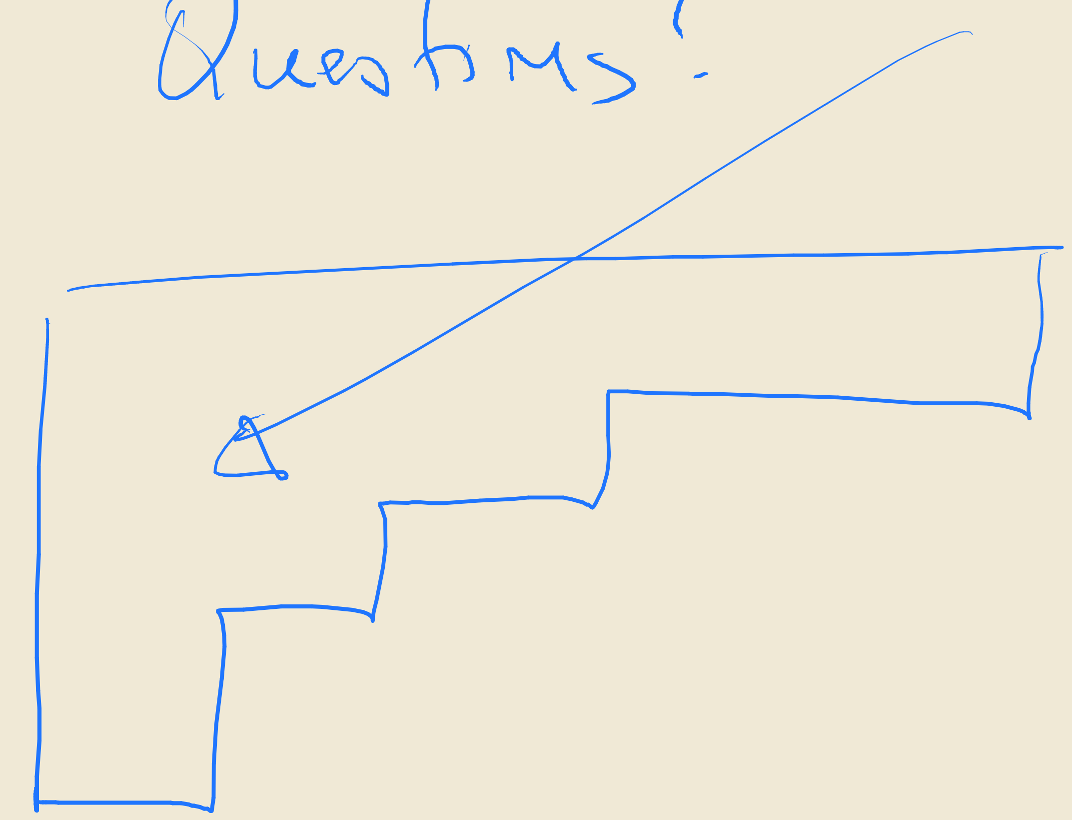
$W = (w^i_j)_{\substack{i \le n \\ j \le N}}$

$\emptyset \xleftarrow{w^1} P_1 \xleftarrow{w^2} P_2 \xleftarrow{w^3} P_3 \leftarrow \cdots \xleftarrow{w^n} P_n$

$\qquad\quad Q_1 \qquad Q_2 \qquad Q_3 \qquad\qquad Q_n$

Want to express $\tilde{x}$ & b as piecewise linear transformations of $\tilde{x}$ & $\tilde{a}$

Questions?

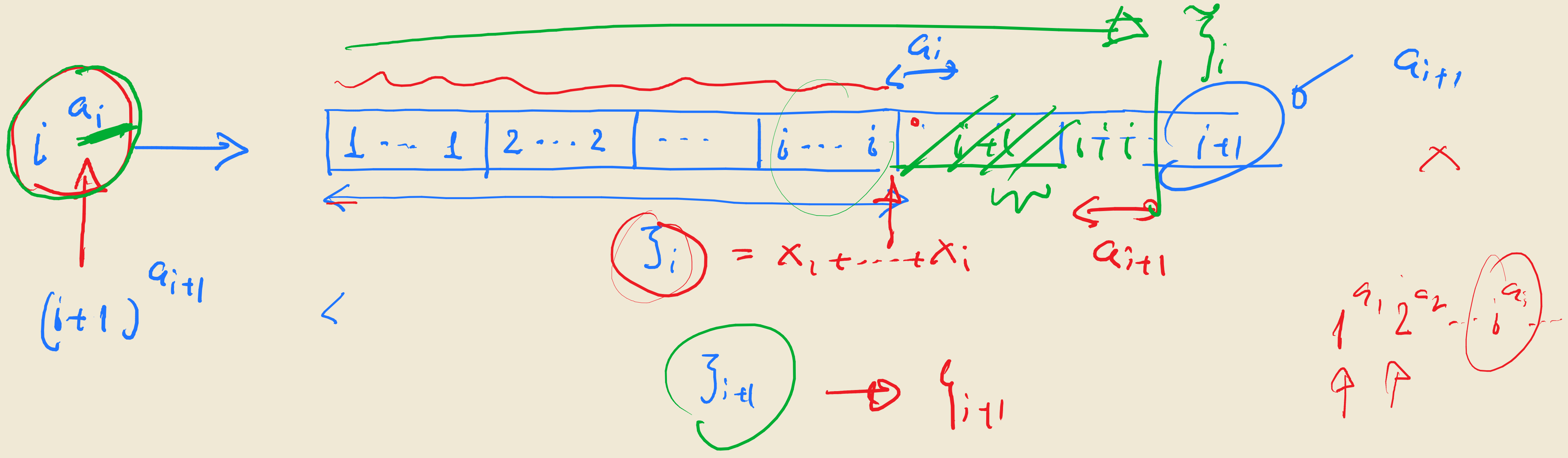


λ_1

λ_2

λ_3

λ_4



Want to see how ξ_i will change (denote the new value $\tilde{\xi}_i$) after insertion of $\underbrace{i \cdots}_{a_i}$

$\boxed{0}$ $\tilde{\xi}_i = \xi_i + a_i$

$\boxed{0}$ $\boxed{\tilde{\xi}_{i+1} = \max(\xi_{i+1}, \tilde{\xi}_i) + a_{i+1}}$ ← Last Passage percolation.

$\boxed{0}$ $b_{i+1} = \min(\tilde{\xi}_i - \xi_i, \xi_{i+1} - \xi_i) = \min(\tilde{\xi}_i, \xi_{i+1}) - \xi_i$

$= \underbrace{a_{i+1} + x_{i+1}} - \tilde{x}_{i+1} = a_{i+1} + (\xi_{i+1} - \xi_i) - (\tilde{\xi}_{i+1} - \tilde{\xi}_i)$

the #(i+1)'s before the insertion + # (i+1)'s inserted

the number of (i+1)'s bumped down.

Clear?

1) the i's that I inserted did not kick out all of the (i+1)'s then

$\tilde{\xi}_{i+1} = \xi_{i+1} + a_{i+1}$

2) the i's that I inserted kicked out ALL (i+1)'s ⇒

$\tilde{\xi}_{i+1} = \tilde{\xi}_i + a_{i+1}$

geometric lifting &

matrix formulation

$(\max, +) \longrightarrow (+, \times)$

Kirillov's geometric lifting of RSK

Previous relations lift to $\zeta_i := x_1 \cdots x_i$

polymers

Markov property

- $\tilde{\zeta}_{i+1} = a_{i+1} \cdot (\tilde{\zeta}_i + \zeta_{i+1})$ $\qquad \tilde{\zeta}_{i+1} = \max(\tilde{\zeta}_i, \zeta_{i+1}) + a_{i+1}$
- $b_{i+1} = a_{i+1} \dfrac{\zeta_{i+1} \cdot \tilde{\zeta}_i}{\zeta_i \quad \tilde{\zeta}_{i+1}}$

Noumi-Yamada from integrability.

Or equivalent

$j \geq 1$

$$\frac{1}{a_j} + \frac{1}{x_{j+1}} = \frac{1}{\tilde{x}_j} + \frac{1}{b_{j+1}}$$

Discrete Toda systems.

KdV.

$$
\begin{pmatrix}
a_1^{-1} & 1 & & & 0 \\
 & a_2^{-1} & 1 & & \\
 & & \ddots & \ddots & \\
0 & & & \ddots & 1 \\
 & & & & a_N^{-1}
\end{pmatrix}
\cdot
\begin{pmatrix}
x_1^{-1} & 1 & & & 0 \\
 & x_2^{-1} & 1 & & \\
 & & \ddots & \ddots & \\
 & & & \ddots & 1 \\
0 & & & & x_N^{-1}
\end{pmatrix}
=
$$

input

$$
\begin{pmatrix}
\tilde{x}_1^{-1} & 1 & & & 0 \\
 & \tilde{x}_2^{-1} & 1 & & \\
 & & \ddots & \ddots & \\
 & & & \ddots & 1 \\
0 & & & & \tilde{x}_N^{-1}
\end{pmatrix}
\cdot
\left(\begin{array}{c|cccc}
1 & 0 & \cdots & & \\
\hline
0 & b_2^{-1} & 1 & & 0 \\
\vdots & & b_3^{-1} & 1 & \\
\vdots & & & \ddots & \ddots \\
 & 0 & & \ddots & b_N^{-1}
\end{array}\right)
$$

output

elementary symmetric function

$$
E(a_1^{-1}, \dots, a_N^{-1})\, E(x_1^{-1}, \dots, x_N^{-1}) = \underbrace{E(\tilde{x}_1^{-1} \cdots \tilde{x}_N^{-1})}\, \underbrace{E_2(b_2^{-1}, \dots, b_N^{-1})}
$$

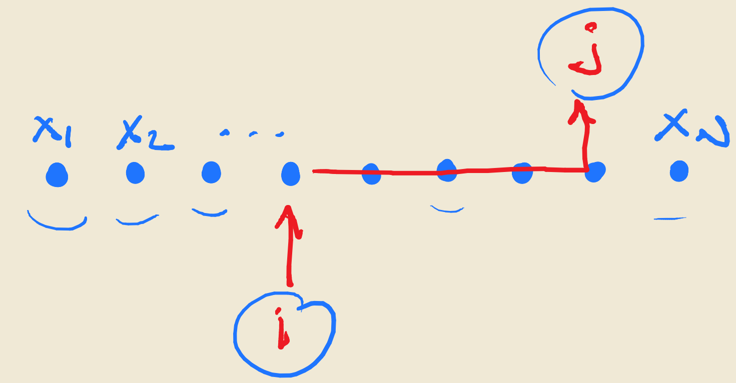
complete symmetric functions

$$H(x_1,\dots,x_N) := D\,E(x_1^{-1},\dots,x_N^{-1})^{-1}D^{-1}$$

$$D = \mathrm{diag}\left((-1)^{i-1}\right)_{i\ge 1}$$

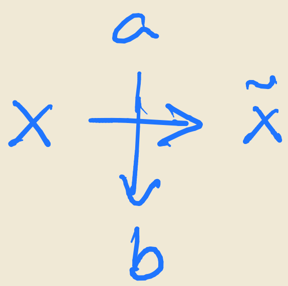
H-matrix formulation; We wrote the gRSK recursions as

$$E(a_1^{-1}\cdots a_N^{-1})\,E(x_1^{-1}\cdots x_N^{-1}) = E(\tilde{x}_1^{-1}\cdots \tilde{x}_N^{-1})\,E_L(b_2^{-1}\cdots b_N^{-1}).$$

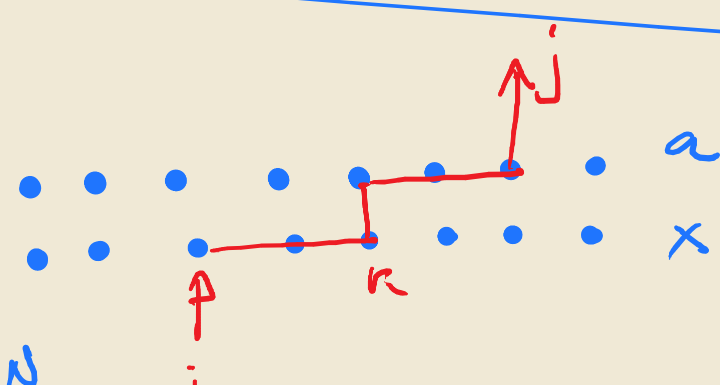
$H(x_1,\dots,x_N)_{i,j} =$  $= \underbrace{x_i\cdots x_j \, 1_{j\ge i}}$

$\parallel$

the total weight of an up-right path from $i \to j$

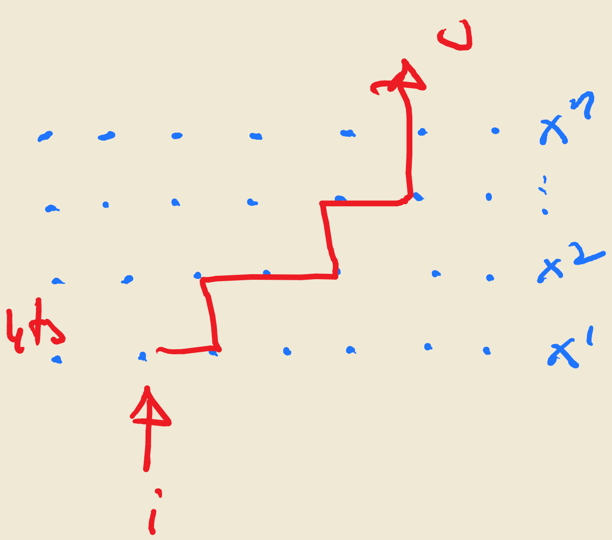
 $\Longleftrightarrow$

$$\left(H(x_1,\dots x_N)\,H(a_1,\dots,a_N)\right)_{ij} = H_2(b_2,\dots,b_N)\,H(\tilde{x}_1,\dots,\tilde{x}_N)$$



$$\sum_{k=1}^{N} H(\vec{x})_{ik}\,H(\vec{a})_{kj}$$

$$\left(H(x^1)\,H(x^2)\cdots H(x^n)\right)_{ij}$$

$\sum_{\pi}\prod$ weights 

Question?

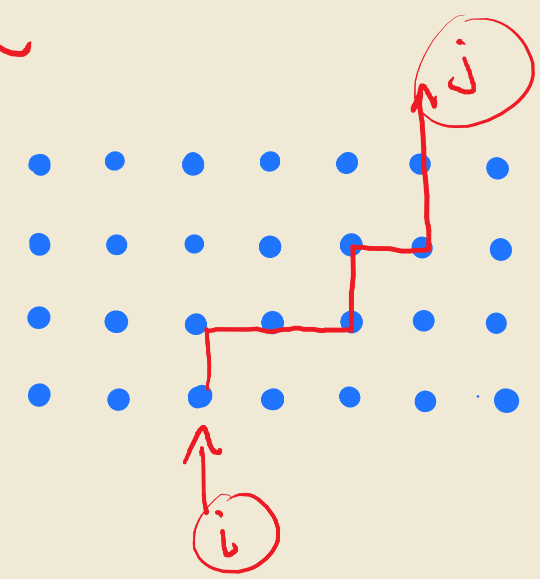
Dynamic RSK

$$X = \left(x^i_j \right)_{\substack{i \le n \\ j \le N}} = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}$$

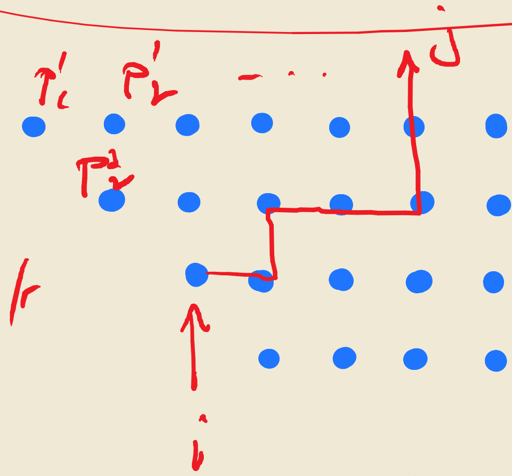
P tableau

$$\left(H(x^1)\, H(x^2) \cdots H(x^n) \right)_{ij} = H_k(p^k)\, H_{k-1}(p^{k-1}) \cdots H_1(p^1)$$

$$\sum_\pi \prod \text{weight}$$



$$= \sum_\pi \prod \text{weight}$$



p_1^1 p_2^1

$\longleftrightarrow$ 1 2 .. 2	
p_2^2	p_3^2

$\vdots$

$$\boxed{H(x^1)\, H(x^2) = H_2(b_2)\, H_1(b_1)}$$

insert x^3.

$$H(x^1)\, H(x^2)\, H(x^3) = H_2(b_2)\, \underbrace{H(b_1)\, H(x^3)}$$

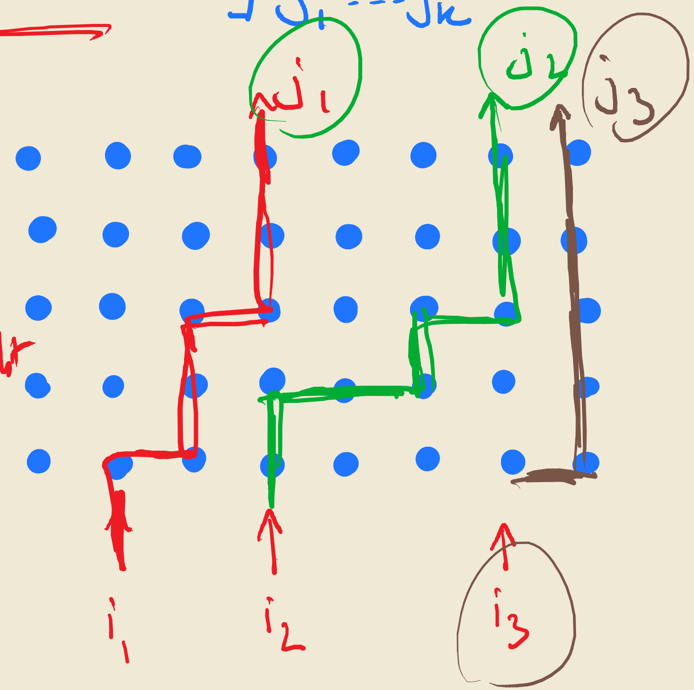
$$= \underbrace{H_2(b_2)\, H_2(b_2')}\, H(p_1)$$

$$= \underbrace{H_3}(p_3)\, H_2(p_2)\, H(p_1)$$

$$\left(\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ \hline 0 & H(p_3) \end{array} \right)$$

Minor determinants

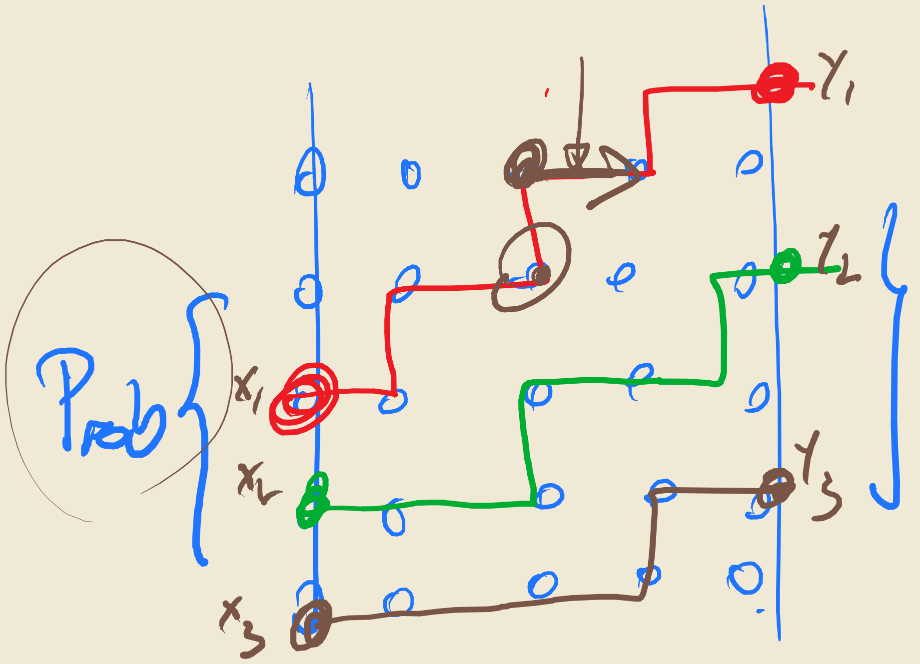
$$\det\left[H(x_1)\,H(x_2)\cdots H(x_k)\right]^{i_1\cdots i_k}_{j_1\cdots j_k} = \sum_{\substack{\pi_1\cdots\pi_k \\ \text{non-intersecting}}} \prod \text{weight}$$



Lindström-Gessel-Viennot theorem

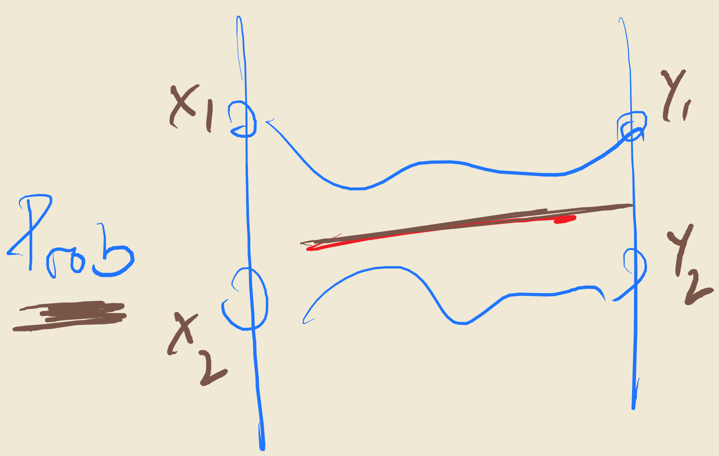
Karlin-McGregor thm

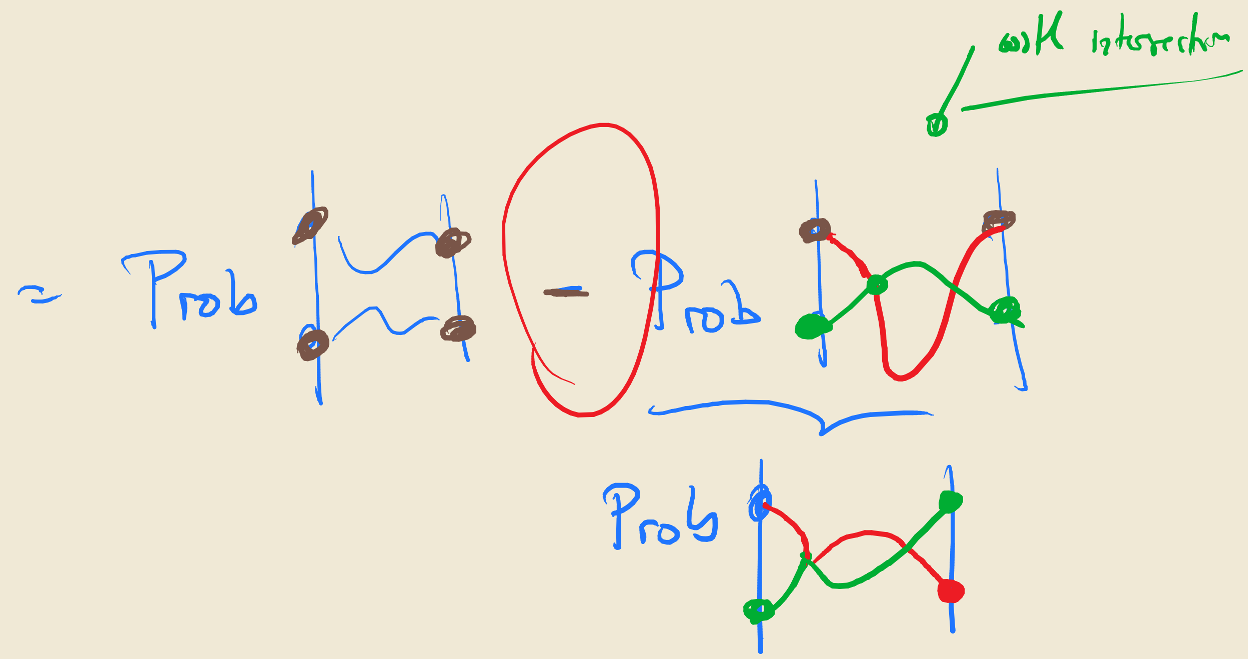
no-intersection!

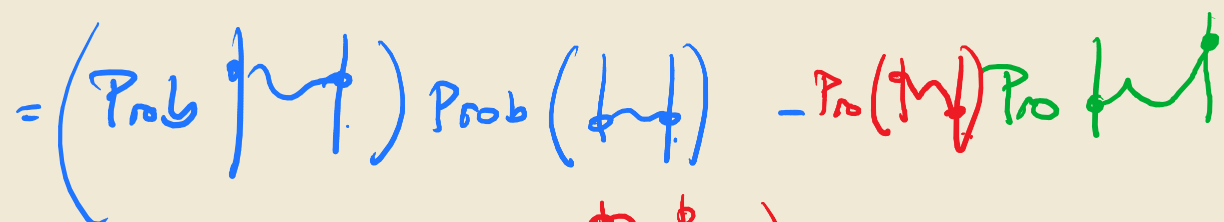


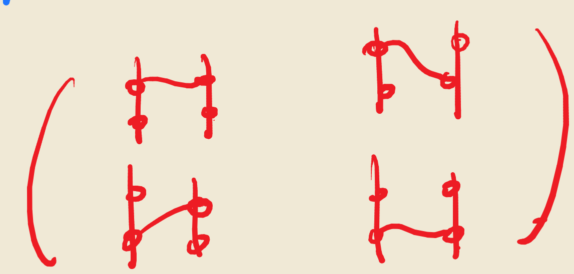
$= \det$ 

Karlin-McGregor thm







det 

Thm (path representation of P tableau)

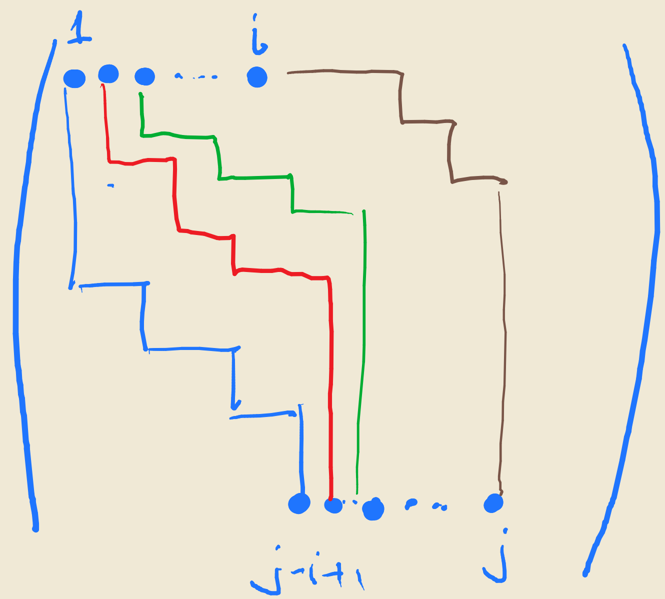
Given a matrix $X = (x^i_j)_{\substack{i \le n \\ j \le N}} = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}$ the equation

$$H(x^1)\, H(x^2) \cdots H(x^n) = H_K(p^K)\, H_{K-1}(p^{K-1}) \cdots H_1(p^1)$$

$$K = n \wedge N$$

has a unique solution $p^i_i = \dfrac{\tau^i_i}{\tau^{i-1}_i}$, $p^i_j = \dfrac{\tau^i_j \, \tau^{i-1}_{j-1}}{\tau^{i-1}_j \, \tau^i_{j-1}}$

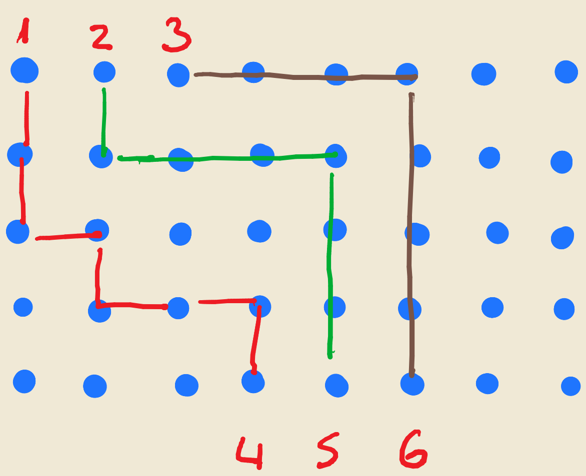
with $\tau^i_j = \sum \prod$

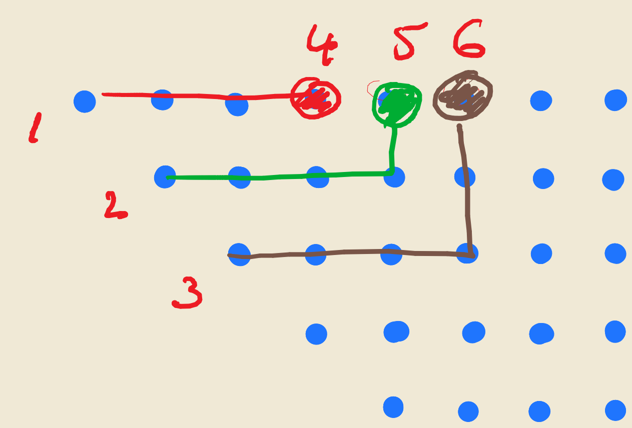


Proof (of the path representation)

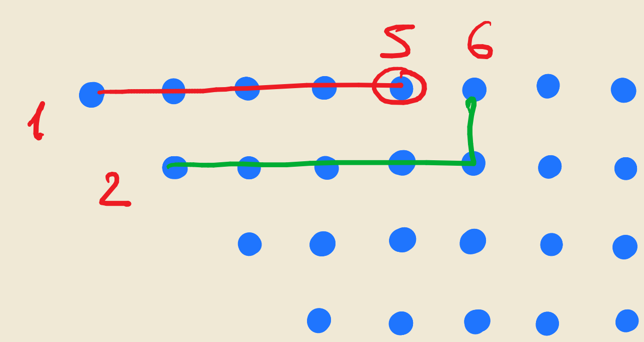
start from $H(x^1)\,H(x^2)\cdots H(x^n) = H_k(p^k)\,H_{k-1}(p^{k-1})\cdots H_1(p^1)$

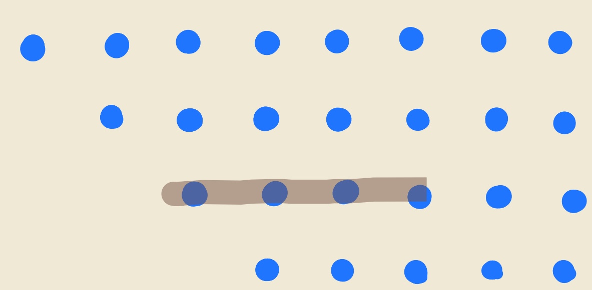
and take $\det\left(\cdots\right)^{1\dots i}_{j-i+1,\dots,j}$ on both sides

$\tau^i_j =$ 

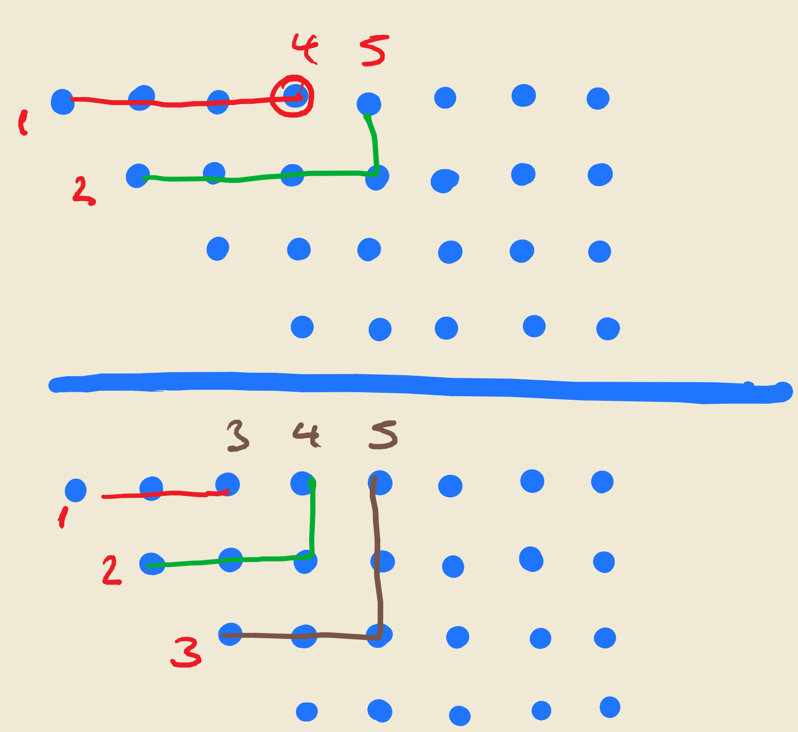
$=$ 

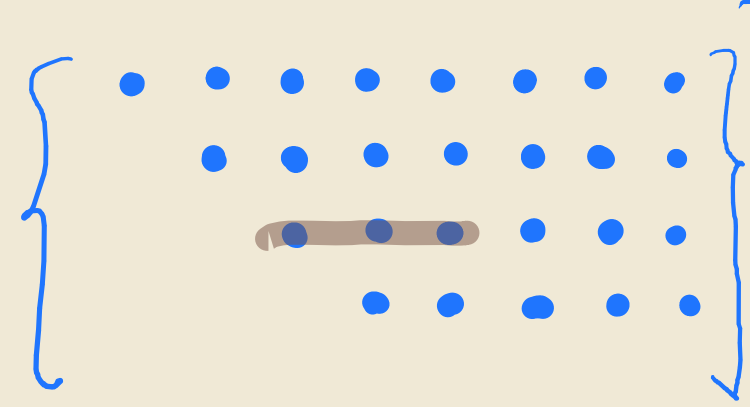
(τ^3_6)

$\tau^{i-1}_j = \tau^2_6$ $=$ 

$\Longrightarrow \quad \dfrac{\tau^i_j}{\tau^{i-1}_j} = \dfrac{\tau^3_6}{\tau^2_6} =$ 

Also,

$\dfrac{\tau^{i-1}_{j-1}}{\tau^i_{j-1}} = \dfrac{\tau^2_5}{\tau^3_5} =$ 

$=$  $^{-1}$