Integrable Probability
Lecture 3

Review of RSK. &

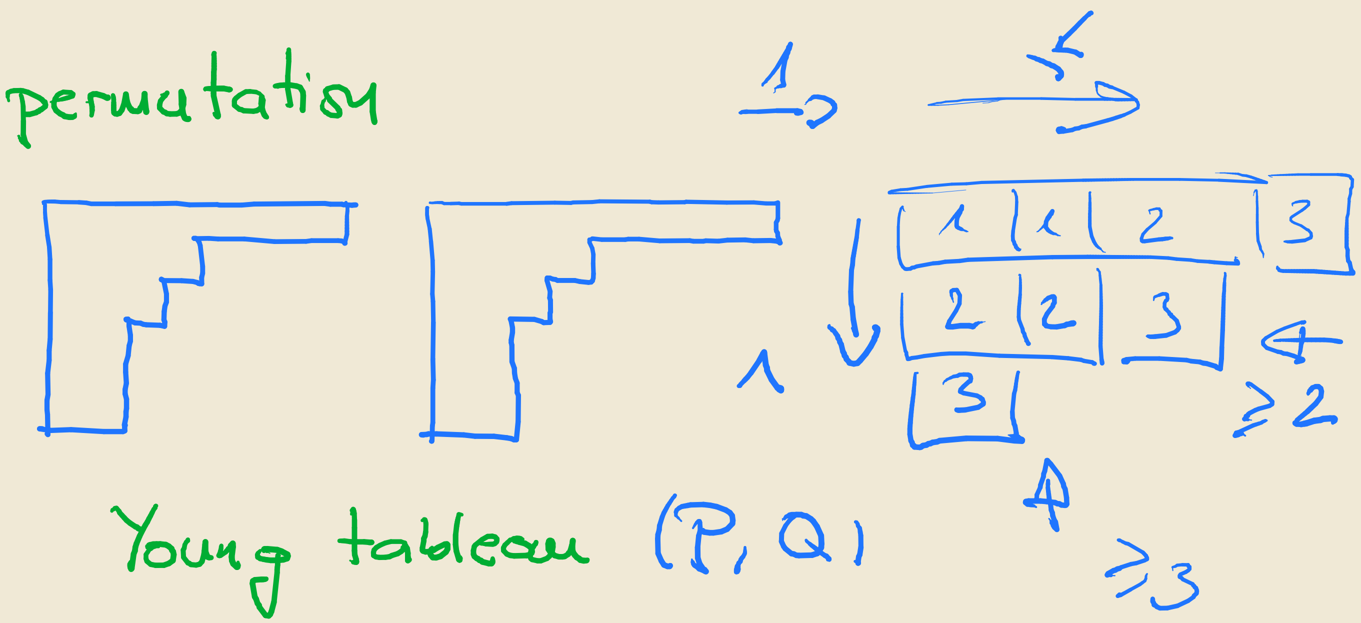
a first example of Integrable Probability :

a solvable Last Passage Percolation

Review of RSK.

RS : bijection between permutation

$\mathfrak{S} \longleftrightarrow$



Young tableau (P, Q)

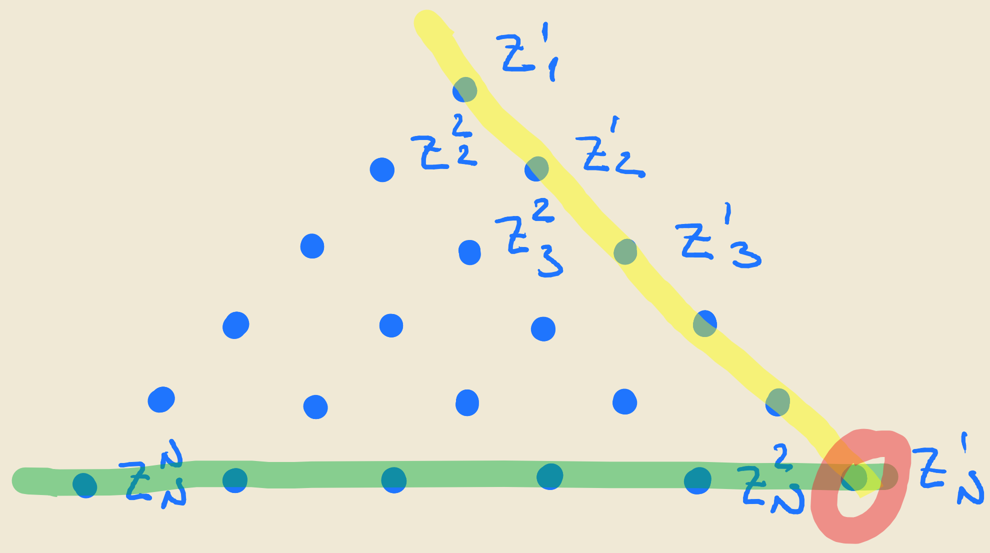
RSK : bijection between matrices

$W = (w_j^i)_{\substack{i \le n \\ j \le N}} \longleftrightarrow$ Gelfand-Tsetlin patterns

$\underbrace{w_j^i \in \mathbb{R}_+}_{\text{input}}$

w_j^i = # of letters j in row i.

Definition: Gelfand-Tsetlin patterns

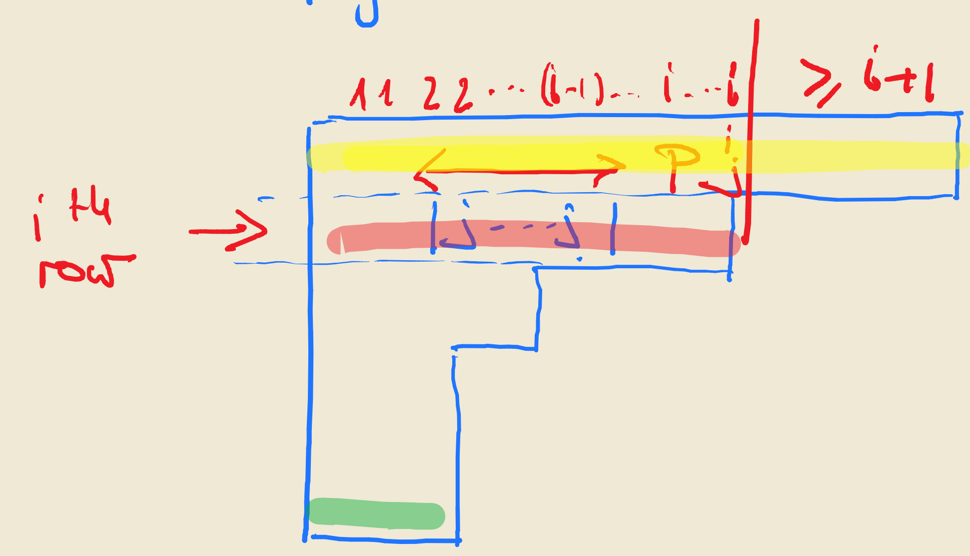


Interlacing triangular patterns

$$z_{j+1}^{i+1} \le z_j^i \le z_j^{i+1}$$

left particle   right particle

Correspondence to Young tableaux p_j^i = # of j's in row i of Young tableau



In GT-variables $z_j^i = p_i^i + p_{i+1}^i + \cdots + p_j^i$ $j \geq i$

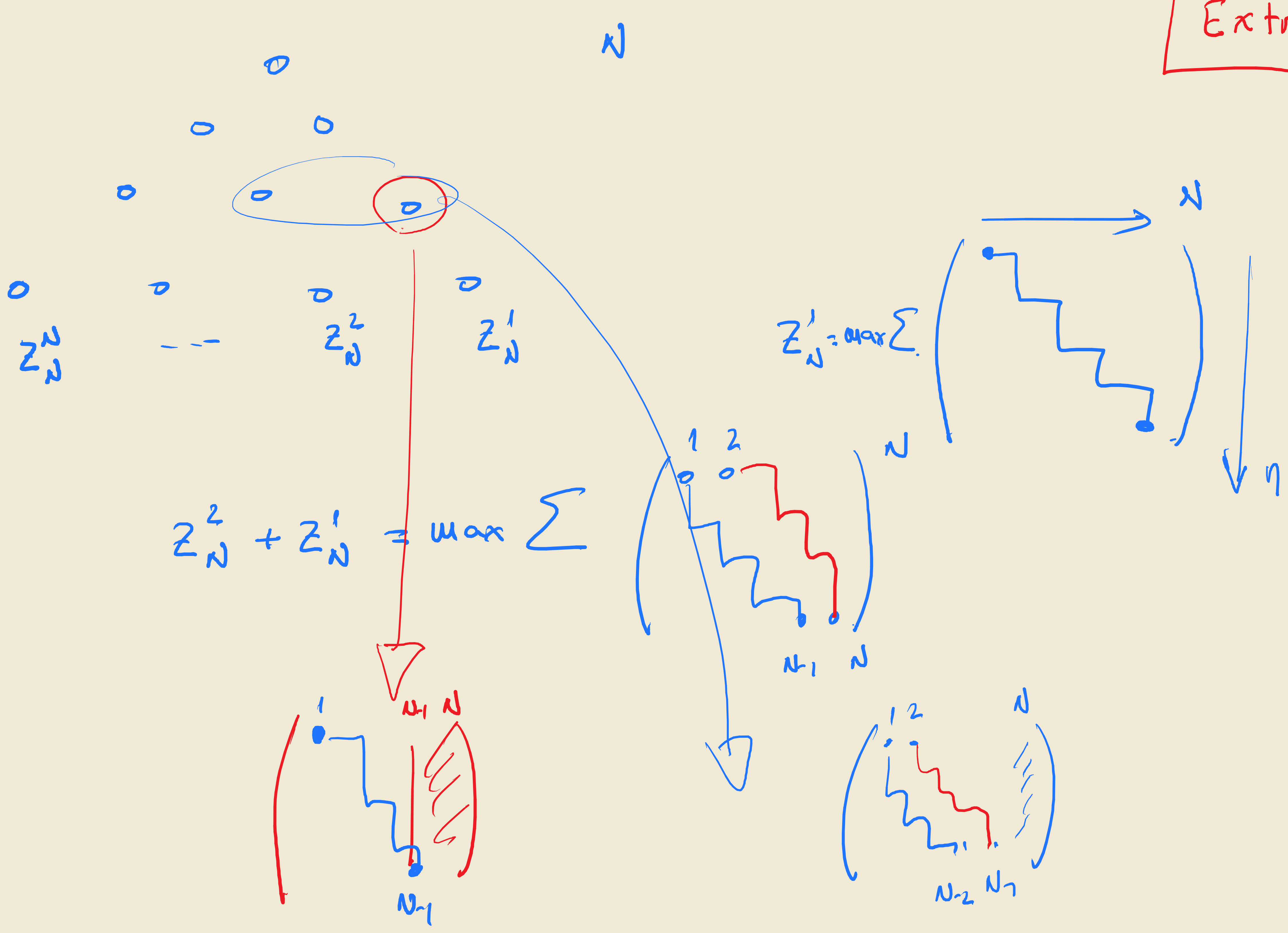
= total # entries up to j in row i

Bottom row of GT = shape of Young tableau

$$z_N^i = p_i^i + \cdots + p_N^i$$

$$= \text{length of } i^{th} \text{ row}$$

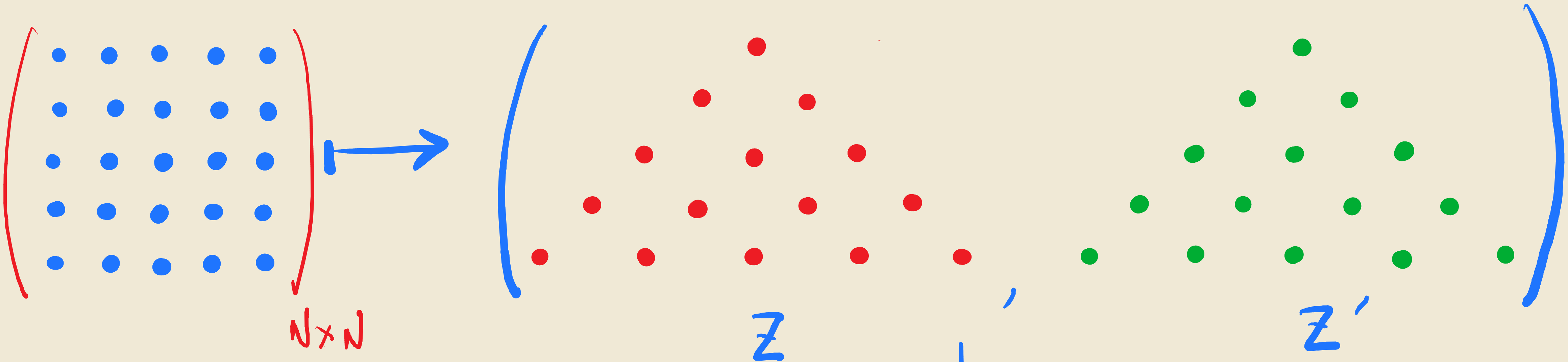
Extras

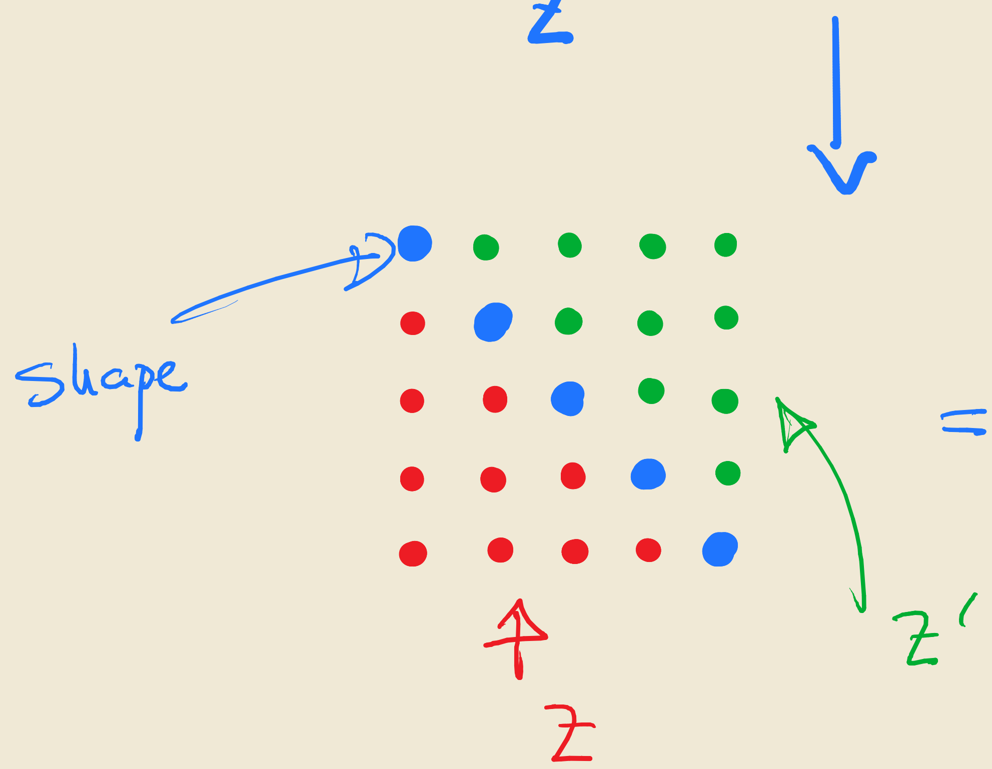


$$z_N^2 + z_N^1 = \max \Sigma$$

RSK as bijection between matrices

$$W = (w^i_j)_{1 \le i,j \le N} \longleftrightarrow (Z, Z') \cong (P, Q)$$





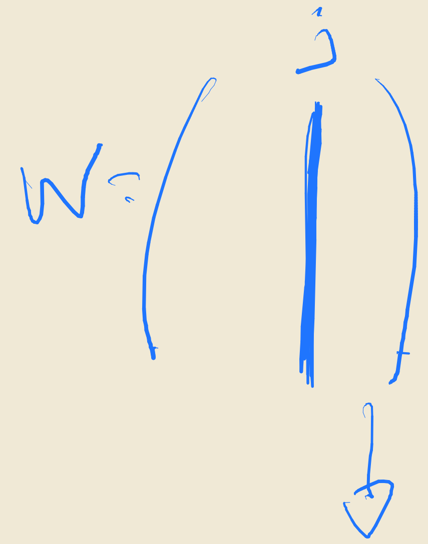
$$=: T = (t^i_j)_{i,j \le N}$$

Extras

$$\left.\begin{aligned} \mathrm{RSK}(W) &= (Z, Z') \cong (P, Q) \\ &\ \| \\ \mathrm{RSK}(W^t) &= (Z', Z) = (Q, P) \end{aligned}\right\}$$

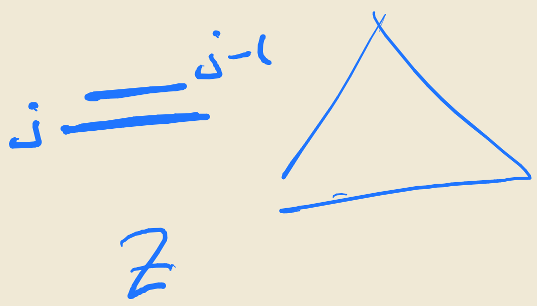
if $W = W^t$ or $\sigma = \sigma^{-1}$ then $P = Q$.

Towards the integrable model: a fundamental property of RSK



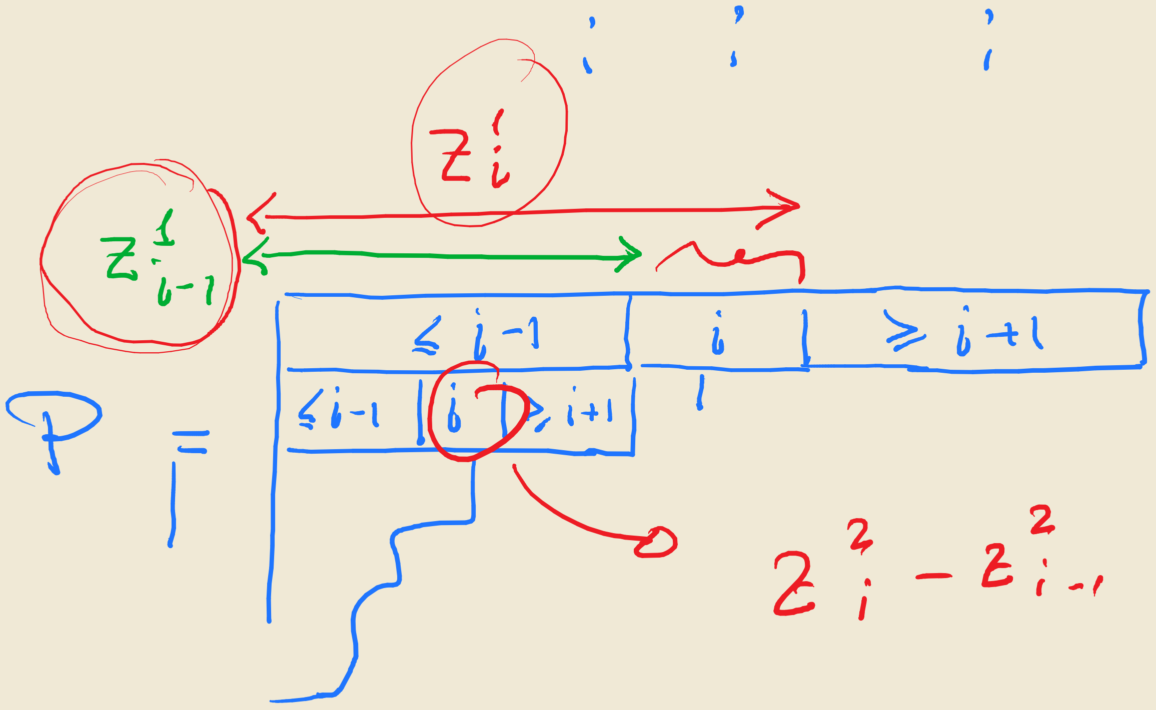
Let $(Z, Z') = RSK(W)$ then

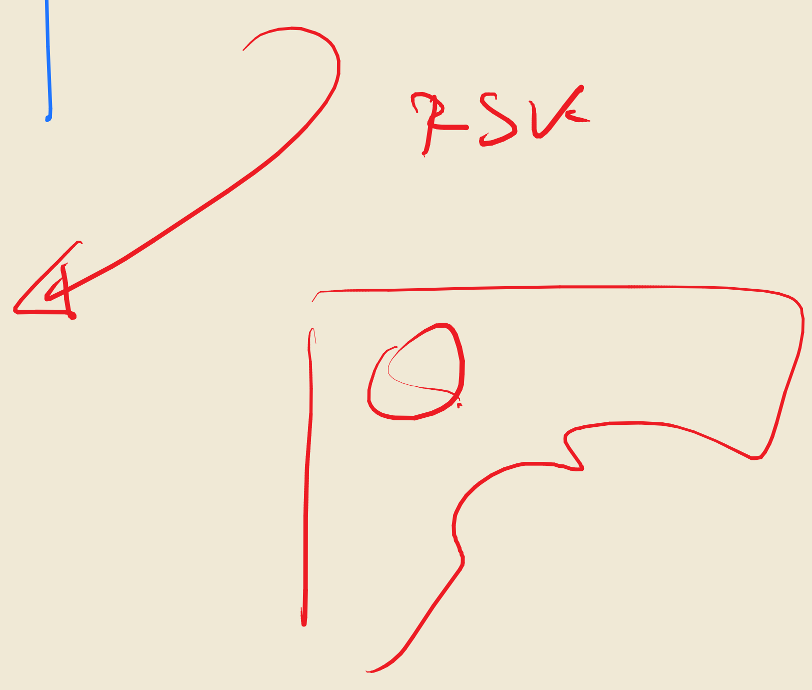
$$\sum_{1 \le i \le n} w_j^i = \sum_{i=1}^{j} z_j^i - \sum_{i=1}^{j-1} z_{j-1}^i$$



Explanation 1: In the combinatorial setting both sides represent the total # i's inserted.

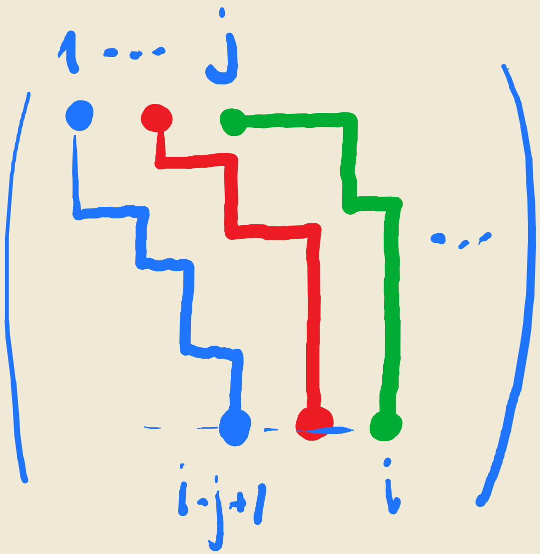
$$(w_j^i) \approx \begin{array}{lllll} 1^{w_1^1} & 2^{w_2^1} & \cdots & i^{w_i^1} & \cdots \\ 1^{w_1^2} & 2^{w_2^2} & \cdots & i^{w_i^2} & \\ \vdots & \vdots & & \vdots & \end{array}$$



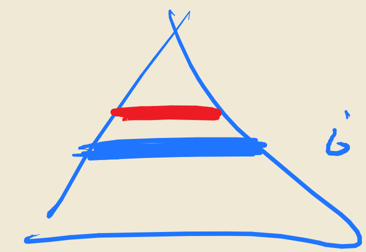


Explanation 2: Path representation

$\max \sum$

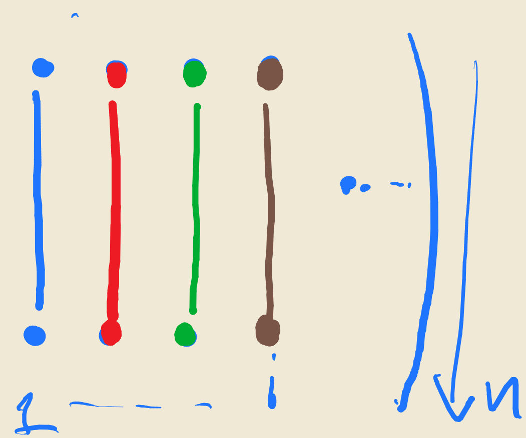


$$= z_i^1 + z_i^2 + \dots + z_i^j$$



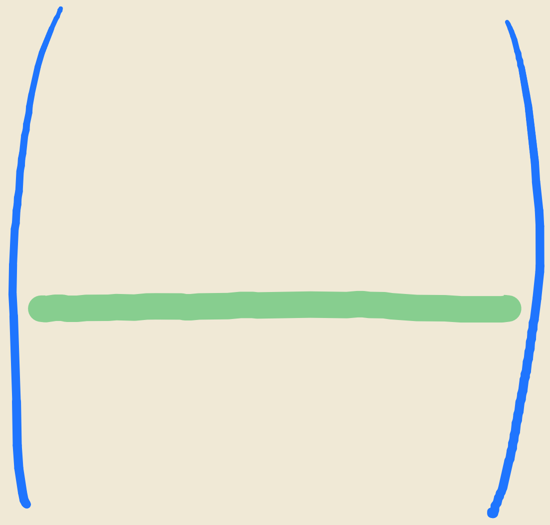
if $j = i$

$\max\limits_{\pi} \sum$



$$= \sum_{\substack{1 \le k \le n \\ 1 \le l \le i}} w_e^k$$

Also,

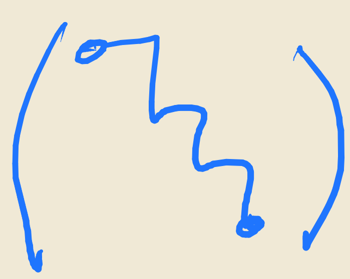
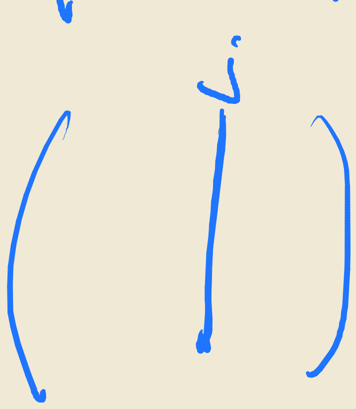


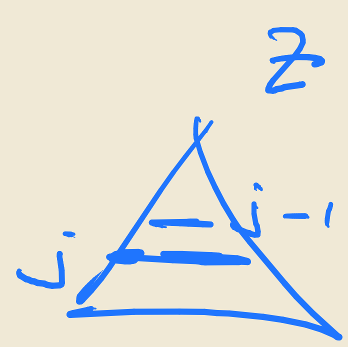
$$= (Z')^1_i + \cdots + (Z')^i_i - (Z')^1_{i-1} - \cdots - Z^{i-1}_{i-1}$$

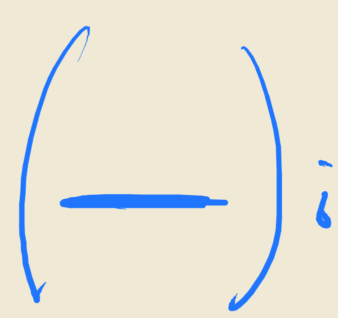
Explanation: $RSK(w^t) = (Z', Z)$

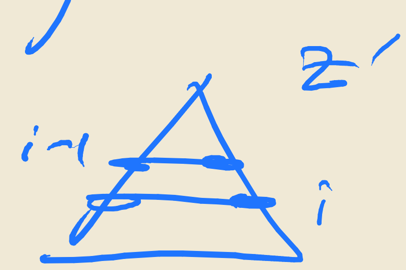
& apply the previous property

Extras

- $RSK(w) = (Z, Z')$
- $RSK(w^t) = (Z', Z)$
- bottom row Z = bottom Z' = shape
-  $= Z_{N1} = Z'_{N1}$
- 

$=$ 

& 

$=$ 

Integrable Model:
Last Passage Percolation with geometric random variables

Thm Let $\mathbf{W} = (\mathbf{W}^i_j)_{i \le i,j}$ with $\{w^i_j\}$ independent

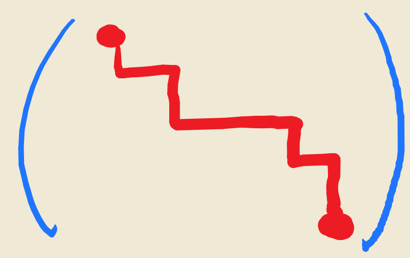
$$\mathbb{P}(\underbrace{\mathbf{W}^i_j = w^i_j}) = (1 - p_i q_j)(p_i q_j)^{w^i_j} \sim \mathrm{Geom}(p_i q_j)$$

then

$$\mathbb{P}(\mathbf{T}_N \le x) = \mathbf{\Gamma}_{p,q} \sum_{\lambda : \lambda_1 \le x} s_\lambda(p)\, s_\lambda(q)$$

with

$$\mathbf{T}_N = \max_{\pi} \sum$$



$$\mathbf{\Gamma}_{p,q} = \prod_{i,j} (1 - p_i q_j)$$

$$s_\lambda(p) = s_{\underset{\sim}{\lambda}}(p_1, \dots, p_N) \quad \text{Schur function}$$

$$\lambda = \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_N \quad \text{"partition"}$$

Extras

$$\sum_{\substack{(\lambda_1, \lambda_2, \dots, \lambda_N) \\ x \ge \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_N}} \underbrace{s_{(\lambda_1, \lambda_2, \dots, \lambda_N)}(p_1, p_2, \dots, p_N)}_{s_\lambda(p)} \cdot \underbrace{s_\lambda(q)}$$

Proof

$$\mathbb{P}(T_N \le x)$$

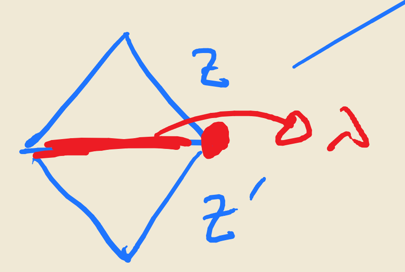
$$= \sum_{W} \mathbb{P}(W)\, \mathbb{1}_{T_N(W) \le x}$$

$$= \prod_{i,j}(1-p_iq_j) \sum_{W} \prod_{i,j} p_i^{w_j^i}\, q_j^{w_j^i}\, \mathbb{1}_{T_N(W)\le x}$$

$$= \prod_{i,j}(1-p_iq_j) \underbrace{\sum_{W}}_{\text{bijection}} \prod_j p_j^{\sum_i w_j^i} \cdot \prod_i q_i^{\sum_j w_j^i}\, \mathbb{1}_{T_N(W)\le x}$$

change of variables

$$= \prod_{i,j}(1-p_iq_j) \sum_{Z,Z'} \prod_j p_j^{|Z^j|-|Z^{j-1}|} \prod_i q_i^{|(Z')^i|-|(Z')^{i-1}|}\, \mathbb{1}_{Z_1^N=(Z')_1^N \le x}$$



$$= \prod_{i,j}(1-p_iq_j) \sum_{x\ge\lambda_1\ge\cdots\ge\lambda_N} \sum_{Z:\, sh(Z)=\lambda} \prod_j p_j^{|Z^j|-|Z^{j-1}|} \sum_{Z':\, sh(Z')=\lambda} \prod_i q_i^{|(Z')^i|-|(Z')^{i-1}|}$$

by one of its definitions

by Schur's definitions.

$$= \prod_{i,j}(1-p_iq_j) \sum_{\substack{\lambda:\, \lambda_1\le x \\ \lambda_1\ge\cdots\ge\lambda_N}} s_\lambda(p)\, s_\lambda(q)$$

Questions?

$$\mathrm{Exp}(x_i+y_j)$$

$$\mathbb{P}(X>u) = e^{-(x_i+y_j)u}$$

$$\prod_{i,j} e^{-x_i}\, e^{-y_j}$$

$$\prod_j e^{-\sum_i x_i} \cdot \prod_i e^{-\sum_j y_j}$$

Extras

for polymer models similar lines lead

$$\mathbb{E}\, e^{-uZ_N} \propto \int_{\mathbb{R}^n_{>0}} \psi_\alpha(x)\, \psi_\beta(x) \prod \frac{dx_i}{x_i}$$

$$\mathbb{P}(w_{ij}\in dx) = \frac{1}{\Gamma(\alpha_i+\beta_j)}\, w_{ij}^{-\alpha_i-\beta_j}\, e^{-\frac{1}{w_{ij}}}\, \frac{dw_{ij}}{w_{ij}}$$

Corollary (Cauchy identity)

$$s_{\lambda_1 \geq \lambda_2 \geq \dots}(\alpha_1, \alpha_2, \dots)$$

$$\sum_{\lambda} s_\lambda(p)\, s_\lambda(q) = \prod_{i,j} \frac{1}{1-p_i q_j}$$

Combinatorial form of Schur

$$s_\lambda(p) = \sum_{T:\, sh(T)=\lambda} \prod p_i^{\#\, i\text{'s in } T}$$

= generating series of Y.T.

$$= \sum_{\substack{Z:\, GT \\ \text{bottom row} = \lambda}} \prod p_i^{|Z^i| - |Z^{i-1}|}$$

Determinantal formula

$$s_\lambda(p) = \frac{\det\left(p_i^{\lambda_j + N - j}\right)_{1 \leq i,j \leq N}}{\det\left(p_i^{N-j}\right)_{1 \leq i,j \leq N}}$$

Determinantal & Asymptotic analysis

I. Basic tools

- Fredholm determinants
- Cauchy - Binet identity
- Sylvester identity

Fredholm determinants

$$\iint \det\begin{pmatrix} K(x_1,x_2) & K(x_1,x_2) \\ K(x_2,x_1) & K(x_2,x_2)\end{pmatrix} dx_1\, dx_2$$

Let $K : L^2(X,\mu) \to L^2(X,\mu)$

$$(Kf)(x) = \int_X K(x,y)\, f(y)\, \mu(dy)$$

trace class i.e. $\int_X |K(x,x)|\,\mu(dx) < \infty$

the Fredholm determinant

$$\det(I+K)_{L^2(X,\mu)} = 1 + \sum_{n=1} \frac{1}{n!} \int\cdots\int_{X^n} \det\big(K(x_i,x_j)\big)_{1\le i,j\le n} \prod_{i=1}^{n} \mu(dx_i)$$

Intuition: Let K a finite matrix. Then

$$\det(I+K) = \prod_{i=1}^{n} (1+\lambda_i) =$$

$$= 1 + \sum_{k=1}^{n} \sum_{1\le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}$$

(Extras)

1st $\sum_{i=1}^{n} \lambda_i = \mathrm{Tr}(K) = \sum_x K(x,x)$

2nd $\sum_{1\le i\ne j\le N} \lambda_i\lambda_j = (\lambda_1+\cdots+\lambda_n)^2 - \sum_{i=1}^{n}\lambda_i^2$

$\sum_{x,y} K(x,x)K(y,y) - \sum_x K^2(x,x)$

$\sum_{x,y} K(x,y)K(y,x)$

$\|$

$$\sum_{x,y} \det\begin{vmatrix} K(x,x) & K(x,y) \\ K(y,x) & K(y,y)\end{vmatrix}$$

$$\det\left(K(x_i,x_j) \right)_{i,j \le n} \xrightarrow{n} \infty$$

$$\downarrow$$

$$\det\left(I + K \right)$$

$L^2(X)$