Lecture 4 :

Determinantal calculus & Asymptotics

Play : Cauchy – Binet identity

Sylvester's identity

From determinants to Fredholm determinants

Steepest descent method & Tracy – Widom

Proposition (Cauchy–Binet or Andreief identity)

Let $\phi_i, \varphi_i \in L^2(X, \mu)$ for $i = 1, \dots, N$

then

$$\frac{1}{N!} \int_{X^N} \det\big(\phi_i(x_j)\big)_{i,j \le N} \det\big(\varphi_i(x_j)\big)_{i,j \le N} \, \mu(dx_1) \cdots \mu(dx_N)$$

$$= \det\left(\int_X \phi_i(x) \varphi_j(x) \, \mu(dx) \right)_{i,j \le N}$$

Proof:

$$\det\big(\phi_i(x_j)\big) = \sum_{\sigma \in S_N} (-1)^\sigma \prod_{i=1}^{N} \phi_i(x_{\sigma(j)})$$

& same for $\varphi_i(x_j)$

$$\Longrightarrow \frac{1}{N!} \int_{X^N} \det(\phi_i(x_j)) \det(\varphi_i(x_j)) \, \mu(dx_1) \cdots \mu(dx_N)$$

$$= \frac{1}{N!} \sum_{\sigma, \tau} (-1)^{\sigma + \tau} \int_{X^N} \prod_{i=1}^{N} \phi_i(x_{\sigma(j)}) \cdot \prod_{i=1}^{N} \varphi_i(x_{\sigma(j)}) \, \mu(dx_1) \cdots \mu(dx_N)$$

Golden rule: Before proceeding look at special examples!

So, $N=2$ & say $\sigma = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $\tau = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

then

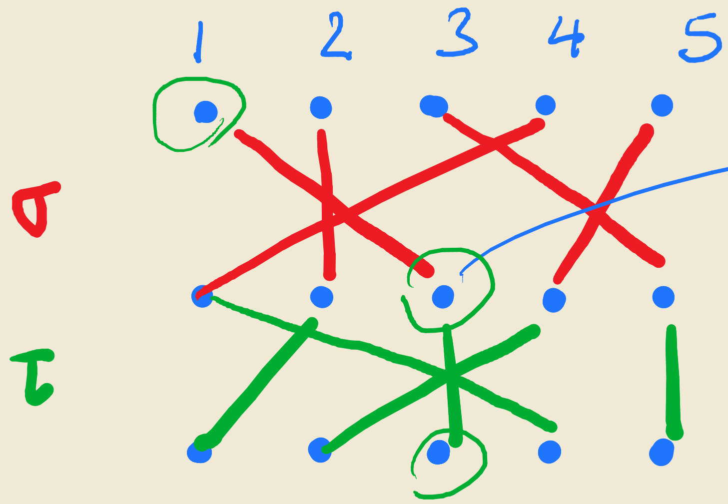
$$\iint \phi_1(x_2)\,\phi_2(x_1) \cdot \varphi_1(x_1)\,\varphi_2(x_2)\, dx_1\, dx_2$$

$$= \int \phi_1(x_2)\,\varphi_2(x_2) \cdot \int \phi_2(x_1)\,\varphi_1(x_1)\, dx_2$$

$$= \int \phi_1(x)\,\varphi_2(x)\, dx \cdot \int \phi_2(x)\,\varphi_1(x)\, dx$$

Observation: things will factorise & then it will be an issue of summing up all permutations σ, τ.

After example(s) --- details !



$\sigma(1) = \tau^{-1}(3)$

$$\begin{array}{ll} \sigma: & 1 \to 3 \\ \tau: & 3 \to 3 \end{array} \qquad \begin{array}{l} 5 \to 4 \\ 4 \to 2 \end{array}$$

$$\int_{X^N} \phi_1(x_{\sigma(1)}) \cdots \phi_N(x_{\sigma(N)}) \cdot \psi_1(x_{\tau(1)}) \cdots \psi_N(x_{\tau(N)})$$

$$\begin{aligned} = \int_{X^N} \quad & \phi_1(x_{\sigma(1)})\, \psi_{\tau^{-1}(\sigma(1))}(x_{\sigma(1)}) \quad dx_{\sigma(1)} \\ \times \; & \phi_2(x_{\sigma(2)})\, \psi_{\tau^{-1}(\sigma(2))}(x_{\sigma(2)}) \quad dx_{\sigma(2)} \\ & \vdots \\ \times \; & \phi_N(x_{\sigma(N)})\, \psi_{\tau^{-1}(\sigma(N))}(x_{\sigma(N)}) \quad dx_{\sigma(N)} \end{aligned}$$

$$= \prod_{i=1}^{N} \int_X \phi_i(x)\, \psi_{\tau^{-1}(\sigma(i))}(x)\, dx$$

... So, we have that

$$\frac{1}{N!}\int_{X^n} \det(\phi_i(x_j))\det(\varphi_i(x_j))\prod_{i=1}^{N}\mu(dx_i)$$

$$= \frac{1}{N!}\sum_{\sigma,\tau\in S_n}(-1)^{\sigma+\tau}\prod_{i=1}^{N}\int_X \phi_i(x)\,\varphi_{\sigma(\tau^{-1}(i))}(x)\,\mu(dx)$$

Set $\sigma\circ\tau^{-1} =: \rho$. Then $(-1)^{\sigma+\tau} = (-1)^{\rho}$

and

$$= \frac{1}{N!}\sum_{\sigma}\sum_{\rho}(-1)^{\rho}\prod_{i=1}^{N}\int_X \phi_i(x)\,\varphi_{\rho(i)}(x)\,\mu(dx)$$

$$= \frac{1}{N!}\sum_{\sigma\in S_N}\det\left(\int_X \phi_i(x)\,\varphi_j(x)\,\mu(dx)\right)_{i,j\leq N}$$

$$= \det\left(\int_X \phi_i(x)\,\varphi_j(x)\,\mu(dx)\right)_{i,j\leq N} \qquad \blacksquare$$

Proposition (Sylvester's identity)

Let $A : L^2(Y,\nu) \longrightarrow L^2(X,\mu)$

$B : L^2(X,\mu) \longrightarrow L^2(Y,\nu)$

be trace class operators. Then

$$\det(I + AB)_{L^2(X,\mu)} = \det(I + BA)_{L^2(X,\mu)}$$

Proof Use the definition of Fredholm determinant

$$\det(I + AB) = 1 + \sum_{n \geq 1} \int_{X^n} \det\left(AB(x_i, x_j)\right)_{i,j \leq n} \prod_{i=1}^{n} dx_i$$

$$= 1 + \sum_{k \geq 1} \sum_{\sigma} (-1)^{\sigma} \int_{X^n} \prod_{i=1}^{n} AB(x_i, x_{\sigma(i)}) \prod_{i=1}^{n} dx_i$$

$$= 1 + \sum_{k \geq 1} \sum_{\sigma} (-1)^{\sigma} \int_{X^n} \prod_{i=1}^{n} \int_{Y} A(x_i, y) B(y, x_{\sigma(i)})\, dy \prod_{i=1}^{n} dx_i$$

$$= 1 + \sum_{k \geq 1} \sum_{\sigma} (-1)^{\sigma} \int_{X^n} \int_{Y^n} \prod_{i=1}^{n} A(x_i, y_i) B(y_i, x_{\sigma(i)}) \prod_{i=1}^{n} dx_i\, dy_i$$

Fubini

$$= 1 + \sum_{k \geq 1} \sum_{\sigma} (-1)^{\sigma} \int_{Y^n} d\vec{y} \int_{X^n} \prod_{i=1}^{n} dx_{\sigma(i)} A(x_{\sigma(i)}, y_{\sigma(i)}) \cdot B(y_i, x_{\sigma(i)})$$

$$= 1 + \sum_{k \geq 1} \sum_{\sigma} (-1)^{\sigma} \int_{Y^n} d\vec{y} \prod_{i=1}^{n} \int_X dx\, B(y_i, x) A(x, y_{\sigma(i)})$$

$$= 1 + \sum_{k \geq 1} \int_{Y^n} d\vec{y} \sum_{\sigma} (-1)^{\sigma} \prod_{i=1}^{n} (BA)(y_i, y_{\sigma(i)})$$

$$= 1 + \sum_{k \geq 1} \int_{Y^n} \det\left(BA(y_i, y_j)\right)_{i,j \leq n} d\vec{y}$$

$$= \det(I + BA)_{L^2(Y,\nu)} \qquad \blacksquare$$

Remark the significance of Sylvester's identity:

it reduces the determinant of a potentially huge matrix to the determinant of a small matrix.

Ex. $a = (a_1, a_2, \ldots)$ $\qquad$ $b = (b_1, b_2, \ldots)$

$$\det(I + a^T b) = \det(I + b\, a^T) = \det\Big(1 + \sum_i a_i b_i\Big)$$

$$\Big\| \qquad 1 + \sum_i a_i b_i$$

$$a^T b = (a_i b_j)_{i,j \geq 1}$$

Proposition (the basic determinantal computation)

Consider a determinantal measure defined as

$$\mu_N(f) = \frac{Z_N(f)}{Z_N}$$

where

$$Z_N(f) = \int_{X^n} \det(\phi_i(x_j))_{i,j\leq n} \det(\psi_i(x_j))_{i,j\leq n} \prod_{i=1}^{n} f(x_i)\, dx_i$$

& $Z_N = Z_N(1)$ is the normalisation or partition function

then $\mu_N(1+g) = \det(I + gK)_{L^2(X)}$

with $$K(x,y) = \sum_{i,j} \psi_i(x)\, [G^{-1}]_{ij}\, \phi_j(y)$$

& $$G_{ij} = \int_X \phi_i(x)\, \psi_j(x)\, dx \qquad \text{(Gram matrix)}$$

Proof $\quad \dfrac{Z_N(1+g)}{Z_N} =$

$$= \frac{\int_{X^N} \det(\phi_i(x_j)) \det(\psi_i(x_j)) \prod (1+g(x_i))}{\int_{X^N} \det(\phi_i(x_j)) \det(\psi_i(x_j))}$$

$$\underset{\text{Binet}}{\overset{\text{Cauchy}}{=}} \frac{\det\left(\int \phi_i(x)\,\psi_j(x)\,(1+g(x))\right)}{\det\left(\int \phi_i(x)\,\psi_j(x)\,dx\right)}$$

$$= \frac{\det\left(G_{ij} + \int \phi_i(x)\,\psi_j(x)\,g(x)\,dx\right)}{\det(G_{ij})}$$

$\dfrac{1}{\det G} = \det G^{-1}$

$\det A \cdot \det B = \det(AB)$

$$= \det\left(\delta_{ij} + \sum_{1\le k\le N} [G^{-1}]_{ik} \int \underbrace{\phi_k(x)}_{A_i(x)}\, \underbrace{\psi_j(x)\, g(x)}_{B_j(x)}\, dx\right)_{i,j\le N}$$

$$= \det\left(\delta_{ij} + \int \sum_{k\le N} [G^{-1}]_{ik}\, \phi_k(\;)\, \psi_j(x)\, g(x)\, dx\right)_{i,j\le N}$$

$$= \det\left(\delta_{ij} + \int A(i,x)\, B(x,j)\, dx\right)_{i,j\le N}$$

where $\quad A(i,x) = \sum_k [G^{-1}]_{ik}\, \phi_k(x)$

$\quad B(x,j) = g(x)\, \psi_j(x)$

$$\underset{\text{Sylvester}}{=} \det\left(I + \sum_i B(x,i)\, A(i,y)\right)_{L^2(X)}$$

$$= \det\left(I + g(x) \sum_{i,k} \psi_i(x)\, [G^{-1}]_{ij}\, \phi_j(y)\right)_{L^2(X)}$$

$\square$

Application : LPP

Prop. $W = (W_{ij})_{i,j \le N}$ with independent entries &

$$\mathbb{P}(W_{ij} = w_{ij}) = (1 - p_i q_j)(p_i q_j)^{w_{ij}}, \quad w_{ij} \in \mathbb{N}$$

then for $\tau_N = \max_{\pi} \sum_{ij \in \pi} (\quad)$

$$\mathbb{P}(\tau_N \le x) = \det\left(I + K_N^{LPP}\right)_{\ell^2(x+N,\, x+N+1,\, \dots)}$$

with

$$K_N^{LPP}(t,s) = \frac{1}{(2\pi i)^2} \oint_{\gamma_1} d\zeta \oint_{\gamma_2} d\eta \, \frac{\eta^s \zeta^t}{1 - \zeta\eta} \prod_{j=1}^{N} \left(\frac{1 - \eta q_j}{\eta - p_j} \right) \cdot \prod_{i=1}^{N} \left(\frac{1 - p_i \zeta}{\zeta - q_i} \right)$$

$\times 1/q$

$$|\gamma_1| = 1 - \varepsilon < 1 + \varepsilon = |\gamma_2|$$

Proof Recall that via RSK we obtained

$$\mathbb{P}(T_N \leq x) = \prod_{i,j} (1 - p_i q_j) \sum_{\lambda : \lambda_1 \leq x} s_\lambda(p) s_\lambda(q)$$

$\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$

Schur function

$$s_\lambda(p) = \left. \frac{\det\left(p_i^{\lambda_j + N - j}\right)}{\det\left(p_i^{N-j}\right)} \right\} =$$

$$= \frac{1}{\Delta(p)} \det\left(p_i^{\lambda_j + N - j}\right)$$

$$\Delta(p) = \det\left(p_i^{N-j}\right) = \prod_{i>j} (p_j - p_i)$$

Vandermonde determinant

So

$$\mathbb{P}(\tau_N \le x) = \frac{\prod_{i,j}(1-p_i q_j)}{\Delta(p)\,\Delta(q)} \sum_{x \geq \lambda_1 \geq \cdots \geq \lambda_N \geq 0} \det\left(p_i^{\lambda_j+N-j}\right) \cdot \det\left(q_i^{\lambda_j+N-j}\right)$$

setting $Z^{-1} := \dfrac{\prod_{ij}(1-p_i q_j)}{\Delta(p)\,\Delta(q)} = \dfrac{1}{\det\left(\frac{1}{1-p_i q_j}\right)}$ Cauchy determinant

& $t_j = \lambda_j + N - j$

then

$$\mathbb{P}(\tau_N \le x) = \frac{1}{Z} \sum_{\substack{\boxed{x+N-1 \geq} t_1 > t_2 > \cdots \\ \cdots > t_N \geq 0}} \det\left(p_i^{t_j}\right) \det\left(q_i^{t_j}\right)$$

symmetry of determinant

$$= \boxed{\frac{1}{N!\,Z}} \sum_{t_1, \ldots, t_N \in \mathbb{N}} \det\left(p_i^{t_j}\right) \det\left(q_j^{t_j}\right) \prod_{i=1}^{N} \mathbb{1}_{[0,x+N-1]}(t_i)$$

this has now the form of a determinantal measure as before with

$$\phi_i(t) = p_i^t \quad \& \quad \psi_i(t) = q_i^t$$

$$\& \quad G_{ij} = \sum_{t \in \mathbb{N}} \phi_i(t)\,\psi_j(t)$$

$$= \sum_{t \in \mathbb{N}} (p_i q_j)^t$$

$$= \frac{1}{1 - p_i q_j}$$

G_{ij} is the Cauchy matrix.

Applying the main determinantal proposition we have

$$\mathbb{P}(T_N \le x) = \det\left(I + gK \right)_{\ell^2(\mathbb{N})}$$

where $g(t) = \mathbb{1}_{[0,x+N-1]}(t)$

& $$K(s,t) = \sum_{i,j} \psi_i(s)\,[G^{-1}]_{ij}\,\phi_j(t)$$

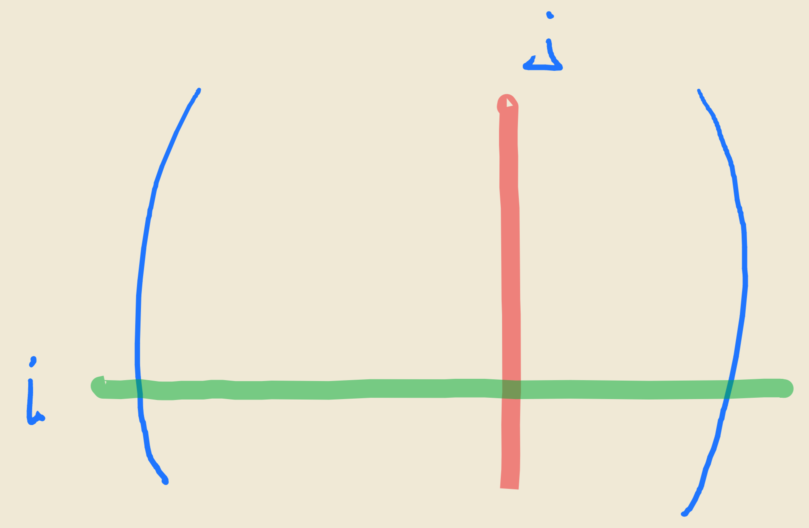
$$= \sum_{i,j} p_i^s\,[G^{-1}]_{ij}\,q_j^t$$

To proceed we need to compute G^{-1} ....

$$G_{ij} = \frac{1}{1-p_i q_j}$$

Cramer's rule $\quad [G^{-1}]_{ij} = \dfrac{(-1)^{i+j}\det G^{ij}}{\det G}$

Notice that the minor



is also a Cauchy matrix with i row & j column removed

So $\det G^{ij}$ can be computed with the same formula as $\det G$

After some cancellation:

$$[G^{-1}]_{ij} = \frac{\prod_{1\le \ell\le N}(1-p_j q_\ell)\prod_{1\le k\le N} 1-p_k q_i}{(1-p_j q_i)\prod_{\ell\neq j}(p_j-p_\ell)\prod_{k\neq i}(q_i-q_k)}$$

and $K(s,t) = \sum_{i,j} q_i^t [G^{-1}]_{ij} p_j^s$

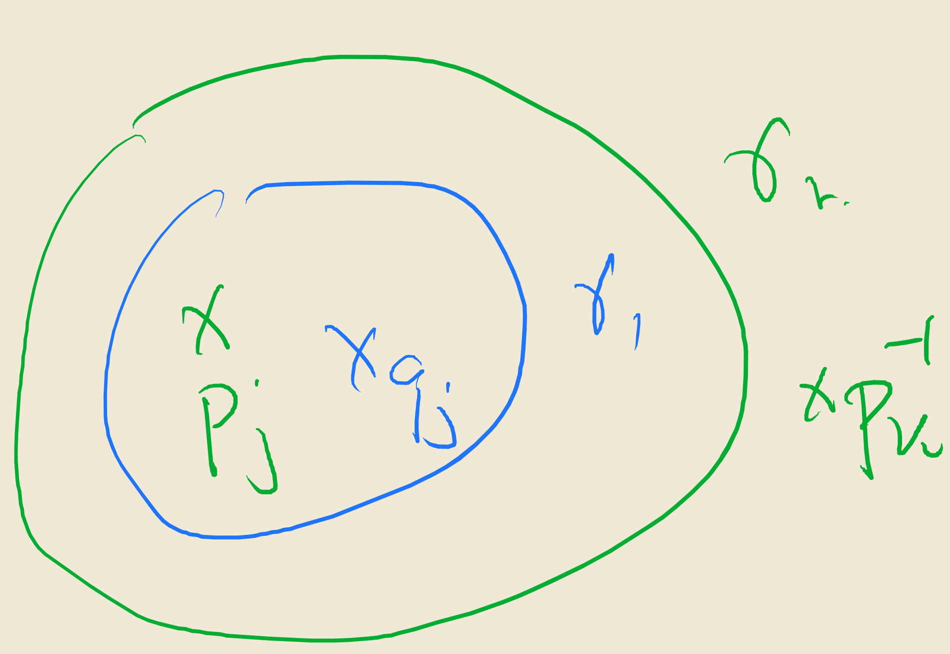
$$K(s,t) = \sum_{1\le i,j\le N} q_i^t p_j^s \frac{\prod_{1\le \ell\le N}(1-p_j q_\ell)\prod_{1\le k\le N} 1-p_k q_i}{(1-p_j q_i)\prod_{\ell\neq j}(p_j-p_\ell)\prod_{k\neq i}(q_i-q_k)}$$

Residue thm. $\quad \dfrac{1}{(2\pi i)^2}\int_{\gamma_1} d\zeta \int_{\gamma_2} d\eta \, \dfrac{\eta^s \zeta^t}{1-\zeta\eta}\prod_{j=1}^N \dfrac{1-\eta q_j}{\eta-p_j}\cdot\dfrac{1-p_j\zeta}{\zeta-q_j}$

Extras

$$\int_{\gamma_1} d\zeta\cdot\zeta^t \sum_k \frac{p_k^s}{1-\zeta p_k}\prod_{j=1}^N \frac{1-p_j\zeta}{\zeta-q_j}\cdot\frac{\prod_{j=1}^N(1-p_k q_j)}{\prod_{j\neq k}(p_k-p_j)}$$

$$\sum_k p_k^s \frac{\prod_{j=1}^N 1-p_k q_j}{\prod_{j\neq k}(p_k-p_j)} \underbrace{\int_{\gamma_1} d\zeta \cdot \frac{\zeta^t}{1-\zeta p_k}\prod_{j=1}^N\frac{1-p_j\zeta}{\zeta-q_j}}$$



$$\sum_\ell \frac{q_\ell^t}{1-q_\ell p_k}\frac{\prod_{j=1}^N 1-p_j q_\ell}{\prod_{\ell\neq j}(q_\ell-q_j)}$$

pay attention to the potential pole at $\zeta = p_k^{-1}$. $\Rightarrow$ there will be no residue if p_k^{-1} is outside the contours.

Example 2 (GUE matrices). (Extras)

$H = (h_{ij})_{i,j \le N}$ hermitian matrix with h_{ij} i.i.d. complex normal variables

$$\frac{h_{ij} + h_{ji}^*}{\sqrt{2}} = N(0,1)$$

The law of ev's $\lambda_1, \dots, \lambda_N$ has density

$$\frac{1}{Z} \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^2 \, e^{-\frac{1}{2}\sum \lambda_j^2} \prod_{i=1}^{N} d\lambda_i$$

$$\frac{1}{Z} \, \Delta(\lambda)\, \Delta(\lambda)\, e^{-\sum \lambda_j^2/2}$$

$$\Delta(\lambda) = \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & & \lambda_2^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_N & & \lambda_N^{N-1} \end{vmatrix} = \det\big(\underbrace{H_j}_{\text{Hermite polynomials}}(\lambda_i)\big)$$

$$G_{ij} = \int H_j(t)\, H_i(t)\, e^{-x^2/2}\, dt = \delta_{ij}$$

$$G^{-1} = I.$$