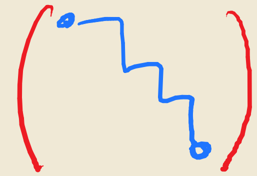
Integrable Probability : Lecture 5

Asymptotic analysis

Introduction to integrable stochastic dynamics

First ingredients

Where we stopped...



Prop. $W = (W_{ij})_{i,j \le n}$ with (W_{ij}) independent

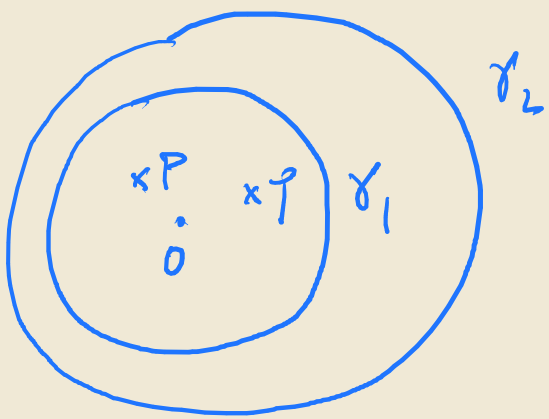
$$\mathbb{P}(W_{ij} = w_{ij}) = (1 - p_i q_j)(p_i q_j)^{w_{ij}} \overset{d}{=} \mathrm{Geom}(p_i q_j)$$

then

$$\mathbb{P}(L_N^{\text{geom}} \le x) = \det\Big(I + \underbrace{\mathbb{1}_{[x+N,\infty)} K_N^{\text{geom.}}}\Big)_{\ell^2(\mathbb{N})}$$

with

$$K_N^{\text{geom.}}(s,t) = \frac{1}{(2\pi i)^2} \int_{\gamma_1} dz \int_{\gamma_2} dw \, \frac{w^s z^t}{1 - zw} \prod_{i=1}^{N} \frac{1 - w q_i}{w - p_i} \cdot \frac{1 - p_i z}{z - q_i}$$



$0 < |\gamma_1| < |\gamma_2| = 1$

& γ_1 contains all p & p's

Corollary (LPP with exponential weights)

$W = (W_{ij})_{i,j \le N}$ with (W_{ij}) independent with

$$\underbrace{\mathbb{P}(W_{ij} = t)}\ = (\alpha_i+\beta_j)^{-1}\, \underbrace{e^{-(\alpha_i+\beta_j)t}} \overset{d}{=} \mathrm{Exp}(\alpha_i+\beta_j)$$

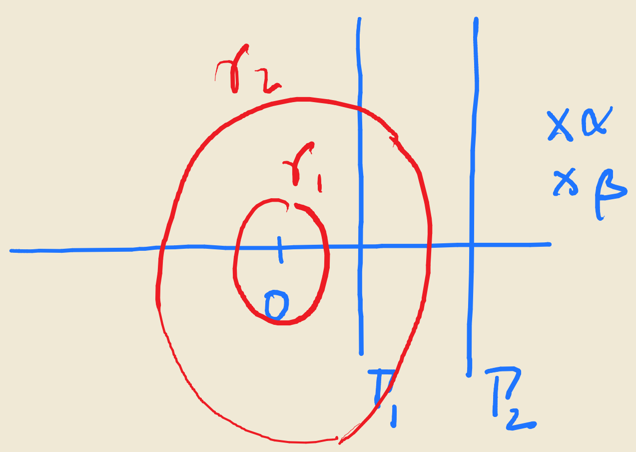
then

$$\underbrace{\mathbb{P}\left(\tau_N^{\exp} \le x\right)} = \underbrace{\det\left(I + \mathbb{1}_{[x+N,\infty)} K_N^{\exp}\right)}_{\underbrace{L^2(\mathbb{R}_+)}}$$

with

$$K_N^{\exp}(s,t) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} dz \int_{\Gamma_2} dy \, \frac{e^{-tz+sy}}{z-y} \prod_i \frac{a_i - y}{\beta_i + y} \cdot \frac{\beta_i + z}{\alpha_i - z}$$

$N \to \infty$



Proof Essentially follows from LPP with geometric variables as

$$\left.\varepsilon^{-1}\, \mathrm{Geom}\left(e^{-\alpha_i \varepsilon} \cdot e^{-\beta_j \varepsilon}\right) \xRightarrow[\varepsilon \downarrow 0]{d} \mathrm{Exp}(\alpha_i + \beta_j)\right\}$$

also change variables in the contour integral as

$$\zeta = e^{-\varepsilon z} \quad \& \quad \eta = e^{-\varepsilon y}$$

$|\zeta| = r \Rightarrow e^{-\varepsilon \mathrm{Re}(z)} = r$

$\mathrm{Re}(z) = \mathrm{const}$

then

$$\underbrace{\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} K^{\text{geom.}}_{N, \varepsilon\alpha, \varepsilon\beta}\left(\frac{t}{\varepsilon}, \frac{s}{\varepsilon}\right)} = K^{\exp}_{N,\alpha,\beta}(t,s)$$

$$\det(I+K) = 1 + \sum_{k \ge 1} \frac{1}{k!} \frac{1}{\varepsilon^k} \underbrace{\sum \dots \sum}_{k\text{-fold}} \det\left(\tfrac{1}{\varepsilon} K^{\text{geom}}\left(t_i/\varepsilon, t_j/\varepsilon\right)\right)$$

$\downarrow$ Rie approx.

$$\int \dots \int \det\left(K^{\exp}(t_i, t_j)\right)$$

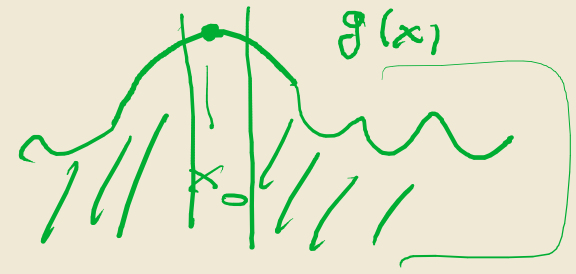
Asymptotics Step 1: Laplace method

["Complex Variables" Ablowitz & Fokas]

Task: compute asymptotics of

$$\lim_{N\to\infty} \int_a^b f(x)\, e^{N g(x)}\, dx \quad \text{as } N \to \infty \qquad f, g \text{ real}$$

$\Rightarrow$ $g(\cdot)$ has a unique max at $x_0 \in (a,b)$ with $g''(x_0) < 0$



Laplace method:

$$\int_a^b f(x)\, e^{N g(x)}\, dx \quad \approx$$

$$g(x) \cong \underline{\underline{g(x_0)}} + \cancel{g'(x_0)}^{\,0}(x-x_0) + \frac{1}{2} g''(x_0)(x-x_0)^2 + o\big((x-x_0)^2\big)$$

$$\approx \quad \underline{\underline{e^{N g(x_0)}}} \int_{(x_0-\varepsilon,\, x_0+\varepsilon)} f(x)\, e^{\frac{N}{2} g''(x_0)(x-x_0)^2 + \text{error}}\, dx$$

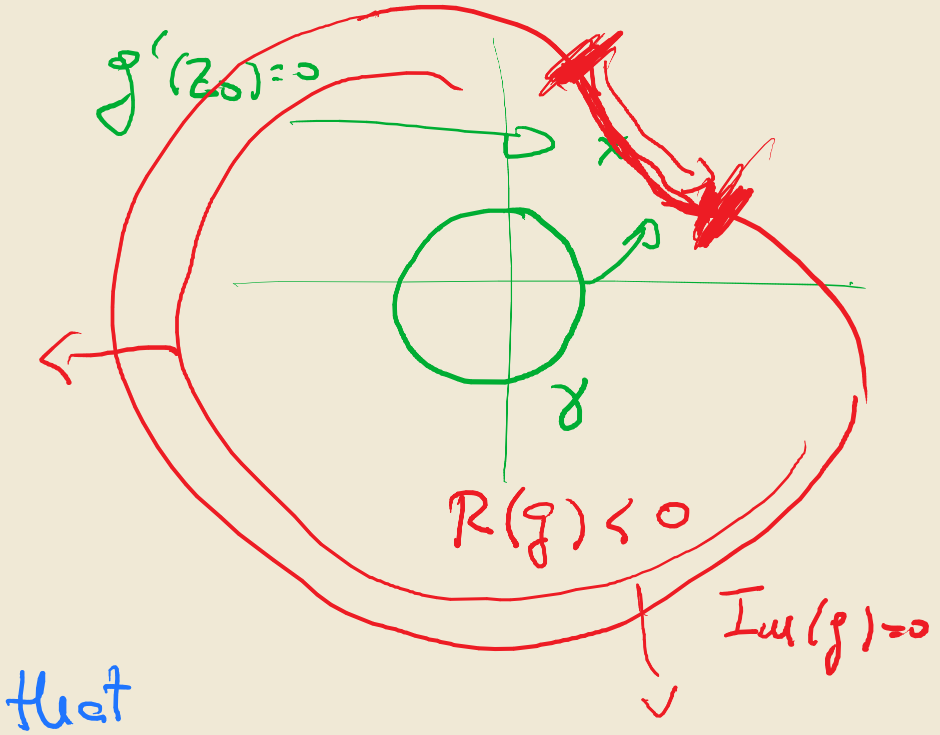
$$\overset{\text{assume}}{\underset{f \text{ cont.}}{\approx}} \quad f(x_0) e^{N g(x_0)} \int_{(x_0-\varepsilon,\, x_0+\varepsilon)} e^{\frac{N}{2} g''(x_0)(x-x_0)^2}\, dx$$

$$= \quad f(x_0) e^{N g(x_0)} \frac{1}{\sqrt{N}} \int_{-\varepsilon}^{\varepsilon} e^{-\frac{N}{2}(-g''(x_0))\, x^2}\, d(x\sqrt{N})$$

$$\pm\varepsilon \to \pm\varepsilon\sqrt{N}$$

$$\underset{N\to\infty}{\approx} \quad f(x_0) \boxed{\frac{e^{N g(x_0)}}{\sqrt{N}}} \underbrace{\int_{-\cancel{\varepsilon\sqrt{N}}\ -\infty}^{\cancel{\varepsilon\sqrt{N}}\ \infty} e^{-\frac{(-g''(x_0))}{2} x^2}\, dx}_{\frac{\sqrt{2\pi}}{\sqrt{-g''(x_0)}}}$$

Asymptotics Step 2: Steepest descent



Task

$$\int_\gamma f(z)\, e^{N g(z)}\, dz \underset{N\to\infty}{\sim} \ ?$$

General principle: • deform the contour γ so that it passes through the critical point of $g(\cdot)$

• then apply Laplace's method.

$$e^{N g(z)} = e^{N \underbrace{Re(g(z))}_{<0} + i N \overset{0}{\cancel{Im(g(z))}}}$$

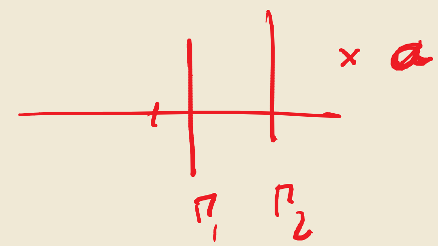
Example of Steepest descent on LPP Kernel

$$\int_\Gamma f(y)\, e^{N \overbrace{\log \frac{a-y}{a+y}}^{g(y)}}\, dy$$

smells like steepest descent

take $\alpha_i = \beta_j = a$ in the previous formula

$$K^{exp}_{N,a}(t,s) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} dz \int_{\Gamma_2} dy \, \frac{e^{-tz+sy}}{z-y} \left(\frac{a-y}{a+y}\right)^N \left(\frac{a+z}{a-z}\right)^N$$



We want to compute the limit as $N \to \infty$ of

$$\mathbb{P}\left(\tau_N^{exp} \le \overbrace{fN + \sigma N^{1/3} x}^{x}\right) =$$

$$= \det\left(I + K^{exp}_N\left(\cdot + fN + \sigma N^{1/3} x, \ \cdot + fN + \sigma N^{1/3} x\right)\right)_{L^2(0,\infty)}$$

this will amount (exercise: fill the details) to computing the limit of

$$\boxed{\underset{\sim}{\sigma} N^{1/3} K_N^{exp}\Big(\underbrace{\sigma N^{1/3} t + f N + \sigma N^{1/3} x}_{t_N}, \underbrace{\sigma N^{1/3} s + f N + \sigma N^{1/3} x}_{s_N}\Big)}$$

Let's start ...

first use $\boxed{\dfrac{1}{z-y} = \int_0^{\infty} e^{-\lambda(z-y)}\, d\lambda}$

& write

$$\boxed{K_N^{exp}(t_N, s_N) = \oint_{\Gamma_1} dz \oint_{\Gamma_2} dy \,\frac{e^{-t_N z + s_N y}}{z-y}\left(\frac{a-y}{a+y}\right)^N \left(\frac{a+z}{a-z}\right)^N}$$

$$= \boxed{\int_0^{\infty} d\lambda} \oint_{\Gamma_1} dz\, e^{-z(t_N+\lambda)} \left(\frac{a+z}{a-z}\right)^N \cdot \oint_{\Gamma_2} dy\, e^{y(s_N+\lambda)} \left(\frac{a-y}{a+y}\right)^N$$

$$=: \int_0^{\infty} d\lambda \oint_{\Gamma_1} dz \,\exp\Big\{ -z(t_N+\lambda) + \underset{\sim}{N}\big[\underbrace{\log(a+z) - \log(a-z)}\big]\Big\}$$

$$\oint_{\Gamma_2} dy \,\exp\Big\{ \text{———} \; s_N, y \; \text{———} \Big\}$$

We will look only into the asymptotics of the z-int.

Recall $t_N = \sigma N^{1/3} t + fN + \sigma N^{1/3} x$

so to match this with the λ we change $\lambda \to \sigma N^{1/3} \lambda$

We will then need to do the asymptotics of

$$\oint_{\Gamma_1} \exp\left\{ N\left[\underbrace{\log(a+z) - \log(a-z) - fz}_{G(z)}\right] - \sigma N^{1/3}(\lambda + t + x) z \right\} dz$$

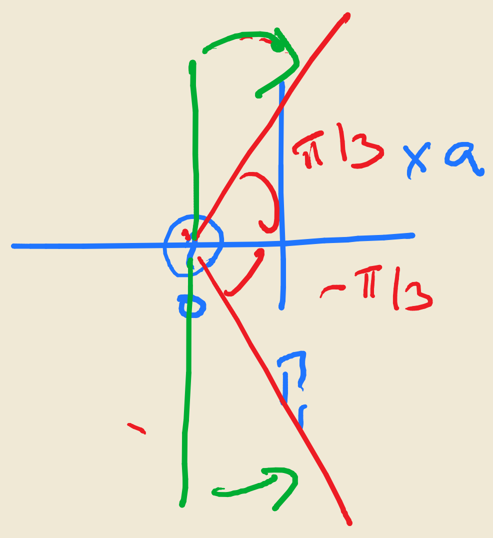
$$G'(z) = \frac{1}{a+z} + \frac{1}{a-z} - f \qquad \text{if } f = \frac{2}{a} \text{ then}$$

$$G'(0) = 0$$

$$G''(z) = -\frac{1}{(a+z)^2} + \frac{1}{(a-z)^2} \qquad G(0) = 0$$

$$G''(0) = 0$$

$$G(z) = \cancelto{0}{G(0)} + \cancelto{0}{G'(0)} z + \frac{1}{2} \cancelto{0}{G''(0)} z^2 + \frac{1}{6} \underbrace{G'''(0)} z^3 + o(z^3)$$



$\operatorname{Im} \tilde{G} = 0$ along the deformed contour.

$\operatorname{Re}(\tilde{G}) < 0$ away from the critical point.

$$\oint_{\Gamma_1} \exp\{ N G(z) - \sigma N^{1/3} (\lambda + t + x) z \} dz$$

$$\simeq \oint_{\tilde{\Gamma}_1 \text{(passes 0)}} \exp\left\{ \underbrace{N \frac{G'''(0)}{6}} z^3 + o(z^3) - \sigma N^{1/3} (\lambda + t + x) z \right\} dz$$

if $z = re^{i\theta} \to z^3 = r^3 e^{3i\theta} = r^3 e^{\pm i\pi} = -r^3$

& choose $\theta = \pm \pi/3$

$$\simeq \int \exp\left\{ N \frac{G'''(0)}{6} z^3 - \sigma N^{1/3} (\lambda + t + x) z \right\} dz$$

$N^{-1/3}$

$$= \int_{B(0, RN^{-1/3}) \cap <} \exp\left\{ N \frac{G'''(0)}{6} z^3 - \sigma N^{1/3} (\lambda + t + x) z \right\} dz$$

change variables $N \frac{G'''(0)}{6} z^3 =: \tilde{z}^3$

$$\propto N^{-1/3} \underbrace{\int_{B(0,R) \cap <_{-\pi/3}^{\pi/3}} \exp\left\{ \tilde{z}^3 - \underbrace{\boxed{\sigma \left(\frac{6}{G'''(0)}\right)^{1/3}}}_{1} (\lambda + t + x) \tilde{z} \right\} d\tilde{z}}_{\downarrow\, R \to \infty}$$

will be taken care by the normalization in front of K.

$$\int_{<} \exp\{ \tilde{z}^3 - (\lambda + t + x) \tilde{z} \} d\tilde{z} = Ai(\lambda + t + x)$$

do the same thing with the y-integral $\leadsto$

$Ai(t+\lambda+y)$

put them back

$$\underset{\text{GUE-kernel}}{\mathrm{Airy}_2} = K(x+t, y+t) = \int_0^\infty Ai(t+\lambda+x)\, Ai(t+\lambda+y)\, d\lambda$$

$\mathbb{R}_+ \to x+t,\ y+t \in (t,\infty)$

then

$$\det\left(I + \mathbb{1}_{[\cdot\cdot]} K_N^{\exp}\right)_{L^2(\mathbb{R}_+)} \longrightarrow \det\left(I + K^{\mathrm{Airy}_2}\right)_{L^2(t,\infty)}$$

$$\boxed{\mathbb{P}\left(L_N^{\exp} \le f N + \sigma N^{1/3} t\right) \xrightarrow[N\to\infty]{} \det\left(I + K^{\mathrm{Airy}_2}\right)_{L^2(t,\infty)}}$$

$= \frac{2}{a}$

$= \mathbb{P}\left(\lambda_1^{GUE} \le \dots\right)$

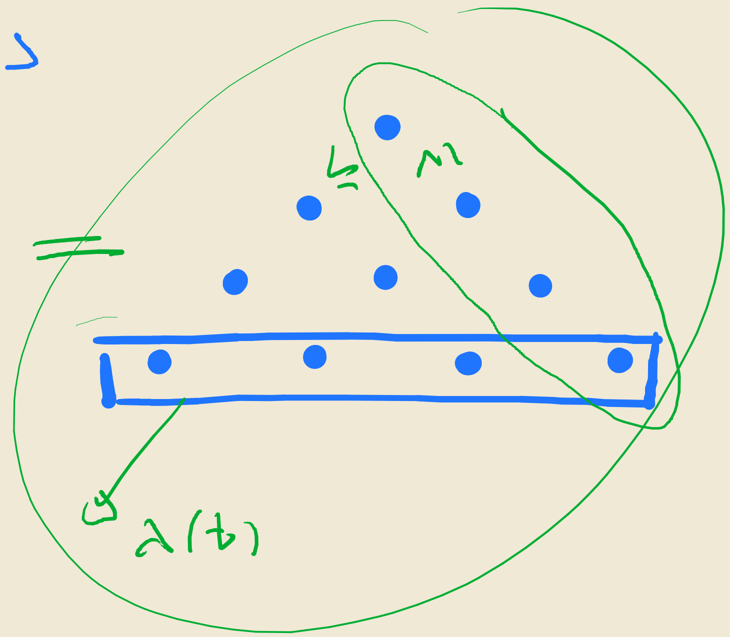
Tracy-Widom GUE distribution

Integrable Stochastic Dynamics

What we will do:

Motivated by combinatorial algorithms s.t. RSK (but not only) we will construct Markovian dynamics on Gelfand-Tsetlin patterns

$Z(t) =$



← Markovian

← this will induce Markovian dynamics here as well

$Z(t)$ will be Markovian by construction

integrability $\Longrightarrow$ $\lambda(t)$ is also Markovian

Markov Functions
or
"When is a function of a Markov process also Markov"?

Thm (Pitman - Rogers, Ann. Prob. 1981)

Let $(Z(t))_{t\geq 1}$ Markov on $\mathcal{Z}$ with transition prob. $\Pi(z,z')$

& a function $\Phi : \mathcal{Z} \to \mathcal{X}$

a kernel $K : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ s.t. $\leftarrow$ Key ingredient

$\to \int K(x,z) \mathbb{1}_{z\in\Phi^{-1}(x)} dz$

- $\forall x \in \mathcal{X} : K(x, \Phi^{-1}(x)) = 1$
- $K\Pi = PK$ $\leftarrow$ intertwining

Then

- $X(t) = \Phi(Z(t))$ is Markov w.r.t. filtration $\mathcal{X}_t = \sigma(X_s : s \leq t)$ starting from $x \in X$

if initial distribution of $Z(\cdot)$ is $\dfrac{K(x,\cdot)}{\int K(x,z)dz}$

- $\forall f : \mathbb{E}[f(Z(t)) \mid X(s), s<t, X(t) = x] = (Kf)(x)$

$X(s) = \Phi(Z(s))$ $\qquad \int K(x,z) f(z) dz$

$\rightsquigarrow$ $K(x,z)$ is the law of $Z(t)$ given the past of $X(t) = \Phi(Z(t))$

Exercise: prove the thm.

Motivation

$\phi(Z(t))$

is this Markov?

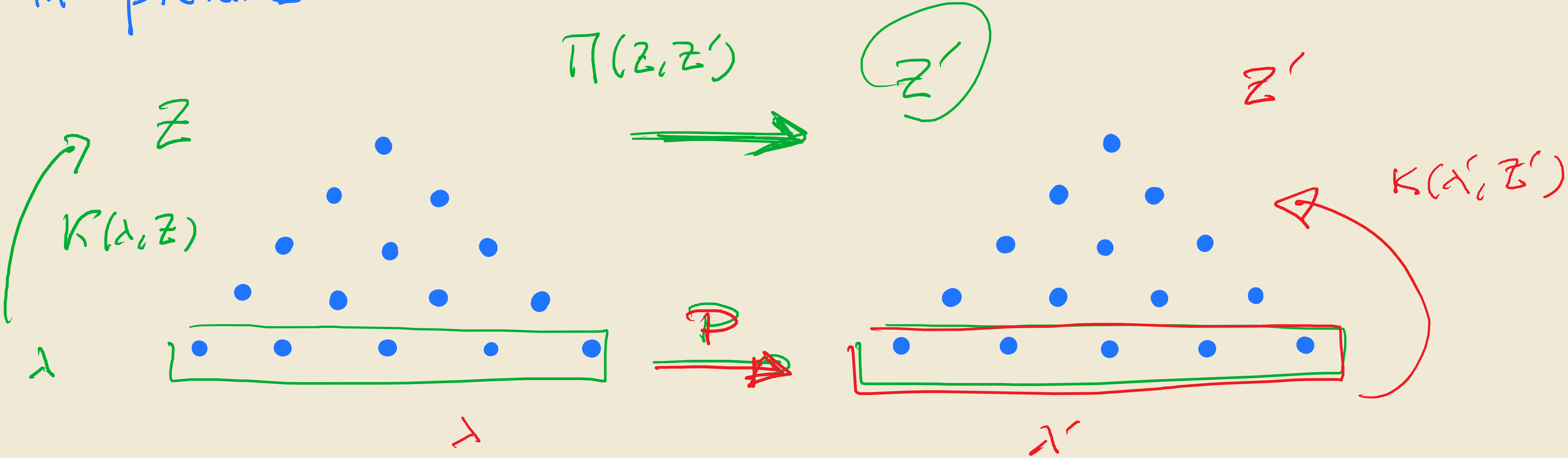
$\beta(t)$

$M_t = \max_{s\leq \cdot} \beta(s)$

$2M_t - \beta(t)$

= Bessel process.

In pictures:



in this case $\Phi(Z)$ = projection of Z to its bottom row.

The difficulty:

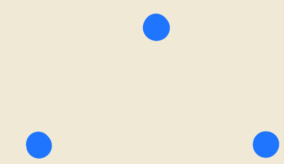
Intertwining

$$\boxed{\Pi K = K P}$$ ⟶ the above diagram commutes

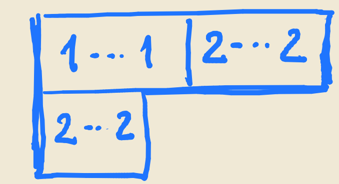
then $(\lambda(t))$ will be Markov with transition kernel P.

RSK induced dynamics

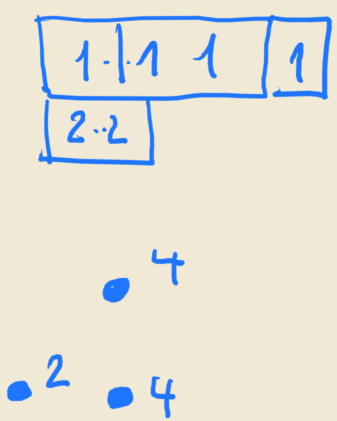
$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ \vdots & \vdots \end{pmatrix} \longrightarrow$$

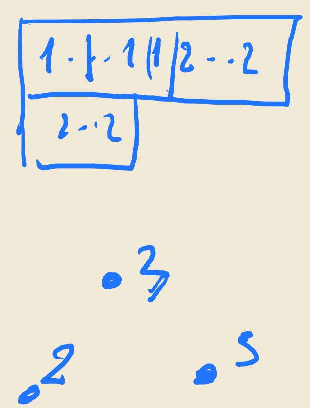


$\cong$



$W^4 = (1,0)$





Ingredients to build dynamics:

- Branching rule, Pieri's rule
- Markov generators, Doob's h-transform

Algebraic $\updownarrow$ probability

$$PK\mathbf{1} = K\Pi\mathbf{1}$$

Branching Rule for Schur

In terms of Gelfand-Tsetlin patterns

$$S_\lambda(x) := S_\lambda(x_1, x_2, \dots, x_n) = \sum_{\substack{Z : GT \\ sh(Z) = \lambda}} \prod_{i=1}^{n} x_i^{|Z^i| - |Z^{i-1}|}$$

with $\quad Z = (Z^i_j)_{i \le j \le n}$

$$sh(Z) = (Z^n_1, Z^n_2, \dots, Z^n_n)$$

$$|Z^i| := \sum_{j=1}^{i} Z^i_j$$

Rewrite

$$S_\lambda(x_1, \dots, x_n) = \sum_{\substack{Z : GT \\ sh(Z) = \lambda}} \prod_{i=1}^{n} Q^i_{i-1}(Z^i, Z^{i-1}; x_i)$$

$$= \sum_{\substack{Z^n = \lambda \\ Z^{n-1} \prec Z^n}} Q^n_{n-1}(Z^n, Z^{n-1}; x_n) \sum_{Z^1, \dots, Z^{n-2}} \prod_{i=1}^{n-1} Q^i_{i-1}(Z^i, Z^{i-1}; x_i)$$

$$= \sum_{\mu : \mu \prec \lambda} Q^n_{n-1}(Z^n, Z^{n-1}; x_n) \, S_\mu(x_1, \dots, x_{n-1})$$