1. a) Define a $\delta$-cover of a subset $X \subset \mathbb{R}^n$ and the Hausdorff $s$-dimensional measure $\mathcal{H}^s(X)$. Define the Hausdorff dimension $\text{dim}_H(X)$ and include an argument for any properties of $\mathcal{H}^s(X)$ that you use. 

b) For $0 < c < 1$ we define the "middle $c$" Cantor set $F_c \subset \mathbb{R}$ by $F_c := \bigcap_{k=0}^{\infty} E_k$ where $E_0 = [0,1], a_0 = 1$ and, for $k \geq 1$, $E_k$ is a union of $2^k$ closed intervals of length $a_k$ obtained by removing from each interval of $E_{k-1}$ the open central interval of length $ca_{k-1}$. Find $a_k$ in terms of $c$ and $k$. Show that $\text{dim}_H(F_c) \leq \log 2/\log(2/(1-c))$. 

c) By defining a suitable measure show that $\text{dim}_H(F_c) = \log 2/\log(2/(1-c))$. 

2. a) Explain the notion of box dimension $\text{dim}_B(X)$ for a bounded subset $X \subset \mathbb{R}^n$. 

b) For $G := \{1/q : q \in \mathbb{N}\}$ show that $\text{dim}_B(G) = 1/2$. 

c) Sketch the set $E := \{z \in \mathbb{C} : 1/|z| \in \mathbb{N}\}$ and show that $\text{dim}_B(E) = 1$. 

[8] [7] [10] [4] [8] [13]
3. a) Define

\[
G := \{ \sum_{n=1}^{\infty} a_n 10^{-n} : \{a_n : n \in \mathbb{N}\} \subset \{0, 1, 2, \ldots, 9\},
\quad \forall k \in \mathbb{N} \ (2k)! < n \leq (2k + 1)! \Rightarrow a_n = 0 \},
\]

\[
G' := \{ \sum_{n=1}^{\infty} a_n 10^{-n} : \{a_n : n \in \mathbb{N}\} \subset \{0, 1, 2, \ldots, 9\},
\quad \forall k \in \mathbb{N} \ (2k - 1)! < n \leq (2k)! \Rightarrow a_n = 0 \}.
\]

Show that \( \dim_H(G) = 0 \).

b) Show that \((x, y) \mapsto x + y\) is a surjective Lipschitz map from \(G \times G'\) to \([0, 1]\).

c) Deduce that \(G \times G' \subset \mathbb{R}^2\) satisfies

\[ \dim_H(G \times G') > \dim_H(G) + \dim_H(G'). \]

State any results from the lectures that you use here.

4. a) What is meant by an Iterated Function System and by the invariant set of an Iterated Function System?

b) What is the Sierpinski Carpet \(G\) in the unit square in the plane? Give an example of an Iterated Function System whose invariant set is \(G\). What are the fixed points of the contractions in your Iterated Function System?

c) For \(t \in [-1, 1]\) define \(E_t := \{x \in [0, 1] : (x, x + t) \in G\}\).

(i) Describe without proof the set \(E_0\).

(ii) Describe without proof the set \(E_{1/2}\).

5. a) Explain the Open Set Condition for an Iterated Function System

b) State without proof a theorem giving the Hausdorff dimension of the invariant set of an Iterated Function System that consists of similarities and satisfies the Open Set Condition. What can you say about the Hausdorff measure of the invariant set in this case?

c) Let \(\Phi := \{\varphi_1, \varphi_2, \varphi_3\}\) and \(\Psi := \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}\) where \(\varphi_j : \mathbb{C} \to \mathbb{C}\) is given by

\[
\varphi_1(z) := z/2, \quad \varphi_2(z) := (z + 1)/2, \quad \varphi_3(z) := (z + i)/2, \quad \varphi_4(z) := (1 + i - z)/2.
\]

Sketch the invariant set for \(\Phi\) and the one for \(\Psi\). Determine whether they satisfy the Open Set Condition and find their Hausdorff dimensions.