MA 3710

THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: APRIL 2010

QUALITATIVE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

Time Allowed: 3 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

1. a) State precisely what is meant by \( f : \mathbb{R}^n \to \mathbb{R}^n \) is locally Lipschitz at a point \( x \).

b) Solve the differential equation \( \dot{x} = 1 + x^2 \) with \( x \in \mathbb{R} \) and initial condition \( x(0) = x_0 \). Show that for all values of \( x_0 \), the solution goes to infinity in finite time, both forwards and backwards.

c) State a theorem on the existence and uniqueness of solutions of an ordinary differential equation, and use it to prove that the solutions you found in (b) above are unique.

d) Give an example to show that \( \dot{x} = f(x) \), with \( x \in \mathbb{R} \) and \( f \) a continuous vector field, can have more than one solution for a given initial condition \( x(0) = x_0 \). Justify your answer.

e) Consider the damped pendulum given by the vector field:

\[
\begin{align*}
\dot{\theta} &= p \\
\dot{p} &= -kp - \sin \theta
\end{align*}
\]

on the cylinder \( S^1 \times \mathbb{R} \) with parameter \( k \geq 0 \). Argue that the solutions to this equation cannot show finite time blow-up for forward time \( (t > 0) \) for any value of the parameter \( k \geq 0 \). What about backward time \( (t < 0) \)?

2. Consider the ideal pendulum on the cylinder $S^1 \times \mathbb{R}$
\[
\dot{\theta} = p \\
\dot{p} = -\sin \theta .
\]

a) Show that the motion conserves $H(\theta, p) = p^2/2 - \cos \theta$. [2]

b) Sketch the level sets \{H(\theta, p) = c\} of $H$ and deduce carefully that every orbit of the ideal pendulum is either periodic or a fixed point or a homoclinic orbit to a fixed point. [Hint: you must show the dynamics within the level sets has the required properties.] [10]

c) Sketch, without calculation, the phase portrait for the pendulum with a small amount of damping.

[3]

d) For an invariant set $\Lambda$ in a flow, state what it means for the set $\Lambda$ to be Liapounov stable.

[3]

e) Prove that for the ideal pendulum (without damping) each periodic orbit is Liapounov stable, but that each homoclinic orbit it NOT Liapounov stable. Determine the Liapounov stability of the two fixed points. Show that the union of the homoclinic orbits and the fixed point at $(\pi, 0)$ is Liapounov stable. [7]

3. a) Suppose that $x_0$ is a fixed point of a differential equation $\dot{x} = f(x)$ where $x \in \mathbb{R}^n$ and $f$ is $C^1$. Define the stable manifold and unstable manifold of $x_0$. State the Stable Manifold Theorem for the case where $x_0$ is a saddle fixed point for a flow in $\mathbb{R}^2$. [5]

b) State carefully what it means for a set $\Lambda$ to be invariant under a flow. Show that the stable and unstable manifolds of a fixed point $x_0$ (in $\mathbb{R}^n$) are invariant sets.

[4]

c) Compute the unstable manifold of the origin to third order in $x$ for the two-dimensional equation:
\[
\dot{x} = x + xy ; \quad \dot{y} = -y + x^2 - xy .
\]
Explain your method carefully. [8]

d) For each of the following statements about the stable and unstable manifolds, $W_s(x_0)$ and $W_u(x_0)$ of a fixed point $x_0$ in a flow in $\mathbb{R}^n$, say whether it is always true, sometimes true, or never true, justifying your answers briefly.

(i) $W_s(x_0)$ and $W_u(x_0)$ intersect in a single point. [8]

(ii) $W_s(x_0) = W_u(x_0)$.

(iii) The sum of the dimensions of $W_s(x_0)$ and $W_u(x_0)$ is $n$.

(iv) $W_s(x_0)$ contains a periodic orbit.
4. a) Let \( \dot{x} = f(x) \) where \( x \in \mathbb{R}^n \) be an \( n \)-dimensional differential equation. State precisely what it means for \( x \) to be a periodic point and \( \Gamma \) to be a periodic orbit of period \( T \) of the flow: Define the Liapounov exponents and the Floquet multipliers of a periodic orbit \( \Gamma \).

b) Consider the van der Pol equation written in Lienard coordinates:

\[
\dot{x} = y - \beta(x^3/3 - x); \quad \dot{y} = -x
\]

when the parameter \( \beta > 0 \) is small.

(i) Write \( H = \frac{1}{2}(x^2 + y^2) \). Compute the time derivative \( \dot{H} \) and explain briefly why an orbit started at a point \((x_0, 0)\) near the origin will return to the \( x \)-axis after a time \( t \approx 2\pi \) at a point \((x_1, 0)\) with \( x_1 \approx x_0 \).

(ii) Show that the change in \( H \) along this piece of the orbit is approximately \( \Delta H = 2\pi\beta(h - h^2/2) \) where \( h = x_0^2/2 \).

(iii) Deduce that for small \( \beta \) there is a periodic orbit passing (approximately) through the point \((2, 0)\) and calculate the (approximate) Floquet multiplier and Liapounov exponent for this orbit.

5. a) Consider the one-dimensional differential equation \( \dot{x} = f(x) \) where \( x \in \mathbb{R} \) and \( f \) is \( C^1 \). State precisely what it means to say that (i) \( x_0 \) is a fixed point; (ii) \( x_0 \) is a hyperbolic fixed point; (iii) \( x_0 \) is an asymptotically stable fixed point. State a sufficient condition for a fixed point to be asymptotically stable.

b) If \( \dot{x} = f(x, \mu) \) is a general one-dimensional differential equation depending on a parameter \( \mu \in \mathbb{R} \), describe, in terms of \( f \) and its derivatives (which you may assume exist), a set of sufficient conditions for the occurrence of a saddle-node bifurcation as the parameter \( \mu \) varies through 0.

c) Consider the differential equation \( \dot{x} = \mu_1 + \mu_2 x - x^2 \) where \( x \in \mathbb{R} \) and \( \mu_1, \mu_2 \in \mathbb{R} \) are parameters.

(i) Describe the bifurcation which occurs when \( \mu_1 \) is varied and \( \mu_2 = 0 \) is kept fixed. Sketch a bifurcation diagram (showing the fixed points and their stability plotted against \( \mu_1 \)). Justify your answers briefly.

(ii) Describe the bifurcation which occurs when \( \mu_2 \) is varied and \( \mu_1 = 0 \) is kept fixed. Sketch a bifurcation diagram (showing the fixed points and their stability plotted against \( \mu_2 \)). Justify your answers briefly.

(iii) Draw a rough sketch of \((\mu_1, \mu_2)\) space indicating the regions in which the differential equation has 0, 1, and 2 fixed points. Explain, with reference to
this sketch, the different types of bifurcation diagram that may be observed if $\mu_1$ and $\mu_2$ are varied together so as to describe a straight line in $(\mu_1, \mu_2)$ space.