Soft Markets: Feedback Effects from Dynamic Hedging

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Outline

- Motivation.
- A simple model.
- Equilibrium and volatility.
- Investor demand as a dynamic programming problem: A more complicated model.
- Numerical example.
- Conclusion.

References

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Introduction

- In the world of derivatives modelling we take market (risk-neutral) dynamics as given and from that calculate derivatives prices:

  Market (model) dynamics => Derivatives prices

- But what if there is a feedback loop:

  Derivatives hedging => Market dynamics

- … what would be the consequence for derivatives prices?

  Derivatives hedging => Market dynamics => Derivatives prices

The Curse of the Hedger

- Suppose banks do (possibly exotic) derivatives transactions with clients and that the banks delta hedge these derivatives but the clients do not.

- Then the delta hedging activity of the banks may affect the dynamics of the underlying:

- If banks are long Gamma then banks need to sell as underlying goes up and buy as the underlying goes down.
  ⇨ the banks' hedging demand will slow market movement
  ⇨ decrease realised volatility.

- If banks are short Gamma then banks need to buy as underlying goes up and sell as the underlying goes down.
  ⇨ the banks' hedging demand will accelerate market movement
  ⇨ increase realised volatility.

- Mechanics work *against* the hedger and *for* the non-hedger.

- The effect of this is a function of market liquidity or more precisely the price elasticity of the market *plus* the net size of the derivatives positions being dynamically hedged.

- The banks will typically be short Gamma at some levels and long at others. This would create volatility skews and smiles.

- The feedback effect has to be incorporated in the pricing of derivatives structures – either implicitly or explicitly.

- Should we concentrate more efforts on quantifying feedback effects rather than attempting to extend our pricing models with stochastic volatility, jumps, etc?

Extreme Examples of Feedback Effects

- Pinning: if an investor succeeds in selling very large option positions to hedgers in an illiquid market it may be possible to pin the stock at the strike at expiration due to very large buy orders below the strike and very large sell orders above. Historical examples (2001) of hedge funds successfully doing this in exchange traded equity options, Avellaneda and Lipkin (2006).

- Market crash: investors buying downside protection from portfolio insurers that delta hedge their position. When market drops portfolio insurers need to sell which will accelerate rapid down moves. The market crash of 1987 was partly blamed on the portfolio insurance strategies sold by Leland-O'Brian-Rubinstein and others. It is not difficult to believe this, as the total outstanding notional under the portfolio insurance programs [i.e. naked delta hedged options (at 15% vol)] was USD 60,000,000,000.

More Relevant Examples of Feedback Effects

- Credit markets: Synthetic CDO mezzanine tranches being delta hedged in the CDS market. The mezz tranche is like a call spread on the expected loss (or index credit spread), hence long convexity for low credit spreads and short convexity for high credit spreads.
  Was this causing credit spreads and credit spread volatility to collapse before August 2007 – and explode after August 2007?

- Long dated FX: Does the hedging of USD/JPY PRDC structures sold to Japanese retail clients affect the dynamics of USD/JPY FX and is this causing the USD/JPY implied volatility skew? Recently: observed drop in USD/JPY and increased volatility is consistent dealers being increasingly short gamma as USD/JPY falls.

- Rates: How does the large amount of outstanding digital (range accrual) trades on ratres or spreads affect yield curve dynamics?

- Hedge funds: Could CPPI structures and unmatched options on hedge funds trigger a hedge fund crash?

- Stock market: Could CPPI structures trigger a real stock market crash?

A Simple Model

- Assume zero rates and dividends.

- Suppose there are two types of active investors: Fundamental investors and hedgers.

- Fundamental investors' demand for the underlying is:

$$a(t) = A \cdot (x(t) - S(t))$$

- … where A is a constant, $S(t)$ is the current spot price of the underlying, and x is the "true" price -- a stochastic process with dynamics:

$$dx(t) = \sigma dW(t) \quad , x(0) = 0$$

- If the difference between the "true" and the spot price goes up the fundamental investor wants to buy more and vice versa.

Hedging Demand

- The hedgers have done derivatives transactions with their clients -- who are assumed not to trade the underlying stock.

- The hedgers do not care about fundamentals or “true” prices, they just need to hedge their derivatives position. [So exactly like the good traders I know].

- The book value of the banks’ derivatives positions is $V(t,S(t))$ -- assumed to come from some model.

- The hedgers’ demand is thus the negative of the delta of their derivatives book value

$$b(t) = -V_S(t,S(t))$$

- …where subscripts denote partial derivatives $V_S = \partial V / \partial S$.

- Note that the hedgers’ demand is only a function of the price of the underlying – the hedger does not care about the level of the fundamental variable.

Equilibrium

- In equilibrium the demand of the fundamental investors and the hedgers must add up to zero (or some arbitrary constant):

$$a(t)+b(t)=0$$
$$\Downarrow$$
$$A\cdot(x(t)-S(t))-V_S(t,S(t))=0$$

- This specifies an implicit equation whose solution for S gives the equilibrium price

$$S=S(t,x(t))$$

- We see that if there is no hedging demand ($b=0$) then the price of the underlying must be equal to the “true” price $S\equiv x$ -- which is why we called it the “true” price.

Volatility

- If we Ito expand the equilibrium condition we get [compressing drift terms]

$$A\cdot(dx(t)-dS(t))-V_{SS}(t,S(t))dS(t)+O(dt)=0$$

- From this we can relate the volatility of the price of the underlying to the volatility of the "true" price:

$$dS(t)=\frac{1}{1+V_{SS}(t,S(t))/A}\sigma dW(t)+O(dt)$$

- We see that

  - Hedgers' Gamma<0 => volatility up

  - Hedgers' Gamma>0 => volatility down

- The magnitude of the effect is determined by the number of fundamental investors to the size of the Gamma of the hedgers: V_{SS} / A.

- Extreme cases:

  - If the hedgers' Gamma goes to +infinity (for example at option expiration) volatility goes to zero. This is the pinning effect described by Avellaneda and Lipkin (2006).

  - If the ratio of Gamma to fundamentalist market share goes to -1 then volatility will go to infinity, ie the market is unstable and we have a potential crash or explosive scenario.

- More interesting case:

  - If the hedgers are gamma long for some s and gamma short for others the described effect will create local volatility and thus volatility smiles.

- Resurrection of local volatility? This shows where local volatility could come from.

Instability?

- We saw that

$$dS(t) = \frac{1}{1+V_{SS}(t,S(t))/A}\sigma dW(t) + O(dt)$$

- If the hedgers are short one standard European call option with expiry T and strike K, and the hedgers are calculating the value of their portfolio on the model $dS(t) = \bar{\sigma} dW(t)$, then

$$V_{SS}(t,S)/A = -\phi\left(\frac{S-K}{\bar{\sigma}\sqrt{T-t}}\right)\frac{1}{\bar{\sigma}\sqrt{T-t}}\frac{1}{A} \underset{\substack{S=K \\ t\to T}}{\longrightarrow} -\infty$$

- Hence, there are spots where the model will blow. This seems problematic.

Smearing

- A way of avoiding the instability is to assume that the option positions in the hedgers' portfolio are distributed across strikes by a continuous distribution:

$$V(t) = E_t^Q[\int (S(T)-K)^+ p(K)dK]$$

- If $p(K) \sim \phi((K-\mu_K)/\sigma_K)$ we have that the ratio of Gamma to fundamental demand is given by

$$1+V_{SS}(t,S)/A = 1-\phi(\frac{S-\mu_K}{v})\frac{1}{v}\frac{1}{A} \quad ,v^2 = \bar{\sigma}^2(T-t)+\sigma_K^2$$

- It follows that the model is stable ($1+V_{SS}/A>0$) as long as

$$A > \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_K}$$

- This idea was introduced in Frey and Stremme (1994).

The One-Period Fundamental Investors: Is A Constant?

- If the fundamental investors know that $b(T+)=0$ then $S(T+)=x(T)$.

- So if the fundamental investor does not trade between now and T, then his wealth at expiration is

$$V(T)=a(t)(x(T)-S(t))+V(t)$$

- …and he maximizes expected utility of the exponential form

$$\sup_{a(t)} E_t[e^{-\gamma V(T)}]$$

- …then the problem of instability is eliminated as fundamental investor demand is

$$a(t)=\frac{x(t)-S(t)}{\gamma\sigma^2(T-t)} \;\Rightarrow\; V_{SS}(t,S(t))/a_S(t)=O(\sqrt{T-t})$$

- Fundamental investor demand increases (uncertainty decreases) as we approach maturity.

The Multi-Period Fundamental Investors

- In the previous section we assumed that the fundamental investor invested as if he was not going to rebalance his portfolio between current time t and some maturity T. It seems more realistic to assume that he can dynamically rebalance his portfolio.

- This leads to an investment problem in which the fundamental investor updates his portfolio dynamically

$$\sup_{\{a(u)\}_{u \geq t}} E_t[e^{-\gamma V(t) - \gamma \int_t^T a(u) dS(u)}] = e^{-\gamma V(t)} \sup_{\{a(u)\}_{u \geq t}} E_t[e^{-\gamma \int_t^T a(u) dS(u)}]$$

- It is easier to attack the discrete version of the problem where the control a is only updated at the discrete dates $0 = t_0 < t_1 < \ldots < t_n = T$:

$$K(t_i) \equiv \sup_{\{a(t_i)\}_{j \geq i}} E_{t_i}[e^{-\gamma \sum_{j=i}^{n-1} a(t_j)(S(t_{j+1}) - S(t_j))}]$$

- We note:

$$K(t_i)=\sup_{a(t_i)} E_{t_i}[e^{-\gamma a(t_i)(S(t_{i+1})-S(t_i))} \sup_{\{a(t_i)\}_{j\geq i+1}} E_{t_{i+1}}[e^{-\gamma \sum_{j=i+1}^{n-1} a(t_j)(S(t_{j+1})-S(t_j))}]]$$
$$=\sup_{a(t_i)} E_{t_i}[e^{-\gamma a(t_i)(S(t_{i+1})-S(t_i))} K(t_{i+1})]$$

- By which we have broken down the multi period problem as a sequence of one-dimensional optimization problems that can be solved backwards and recursively.

- The first order condition $\partial K(t_i)/\partial a(t_i)=0$ gives us that the optimal position $\hat{a}$ needs to solve

$$0=E_{t_i}[S(t_{i+1})e^{-\gamma \hat{a} S(t_{i+1})}K(t_{i+1})]-S(t_i)E_{t_i}[e^{-\gamma \hat{a} S(t_{i+1})}K(t_{i+1})]\equiv L(t_i,\hat{a})-S(t_i)M(t_i,\hat{a})$$

- Note that we assume price taking behaviour here: The fundamental investor takes the observed price as and changes his portfolio holdings accordingly.

- We note that L,M solve

$$0=L_t+\frac{1}{2}\sigma^2 L_{xx} \quad ,L(t_{i+1},a)=S(t_{i+1})e^{-\gamma aS(t_{i+1})}K(t_{i+1})$$
$$0=M_t+\frac{1}{2}\sigma^2 M_{xx} \quad ,M(t_{i+1},a)=e^{-\gamma aS(t_{i+1})}K(t_{i+1})$$

- At the same time equilibrium dictates that $a(t_i)=V_S(t_i,S(t_i))$. Plugging this into the FOC yields that the equilibrium price $\hat{S}$ satisfies

$$0=L(t_i,V_S(t_i,\hat{S}))-\hat{S}M(t_i,V_S(t_i,\hat{S}))$$

- This is solved for each level of x and from this we obtain $\hat{S}=\hat{S}(t,x(t))$ and ready to move one step backward by using $K(t_i)=e^{\gamma\hat{a}\hat{S}}M(t_i,\hat{a}) \quad ,\hat{a}=V_S(t,\hat{S})$.

- The spot volatility can now be found as

$$d\hat{S}(t)=\hat{S}_x(t)\sigma dW(t)+O(dt)$$

- This seems complicated (and it is) but it is a feasible exercise for the toy models like $dx = \sigma dW$ where FD is applicable. Numerical pricing runs in less than 1 second.

- But, it is probably too heavy a methodology to implement for more complicated (MC) models such as multi factor interest rate models.

- The fundamental investors are assumed to know the exact demand (Delta profile) of the hedgers. Can be rectified by adding in measures of random demand (noise traders):

$$\underbrace{a(t)}_{fundamental} - \underbrace{V_S(t, S(t))}_{hedgers} - \underbrace{y(t)}_{noise\,traders} = 0 \quad , dy(t) = -\kappa y(t) dt + \varepsilon dZ(t)$$

- But this will increase the numerical complexity by another order (=> 2F problem).

- Non-price taking behaviour can be incorporated in the solution methodology (Almgren and Chriss (1998)). In fact, the optimal liquidation problem of Almgren and Chriss is very easy to solve with the outlined methodology.

Is the Hedger Loosing Money to the Fundamental Investor?

- Not necessarily. Remember that a hedger that lives in a market with volatility ϑ but hedges and values off a volatility of $\bar{\sigma}$ will have a book value that evolves according to (Dupire (1995)). So a Delta flat hedger book evolves according to

$$dV(t) = \frac{1}{2}(\vartheta(t)^2 - \bar{\sigma}(t)^2)V_{SS}(t)dt$$

- So the hedger will on average not loose money as long as the pricing/hedging volatility $\bar{\sigma}$ is chosen so that

$$E[\frac{1}{2}\int_0^T (\vartheta(t)^2 - \bar{\sigma}(t)^2)V_{SS}(t)dt] \geq 0$$

Numerical Example

- Suppose the hedgers' have entered a 10y swap where they pay their clients a high coupon as long as spot is below a certain strike against receiving a fixed coupon. So each net coupon is

$$\underbrace{-c_1 \min(1, \max(k + \Delta k / 2 - S) / \Delta k)}_{short\ put\ spread} + \underbrace{c_2}_{long\ fixed}$$

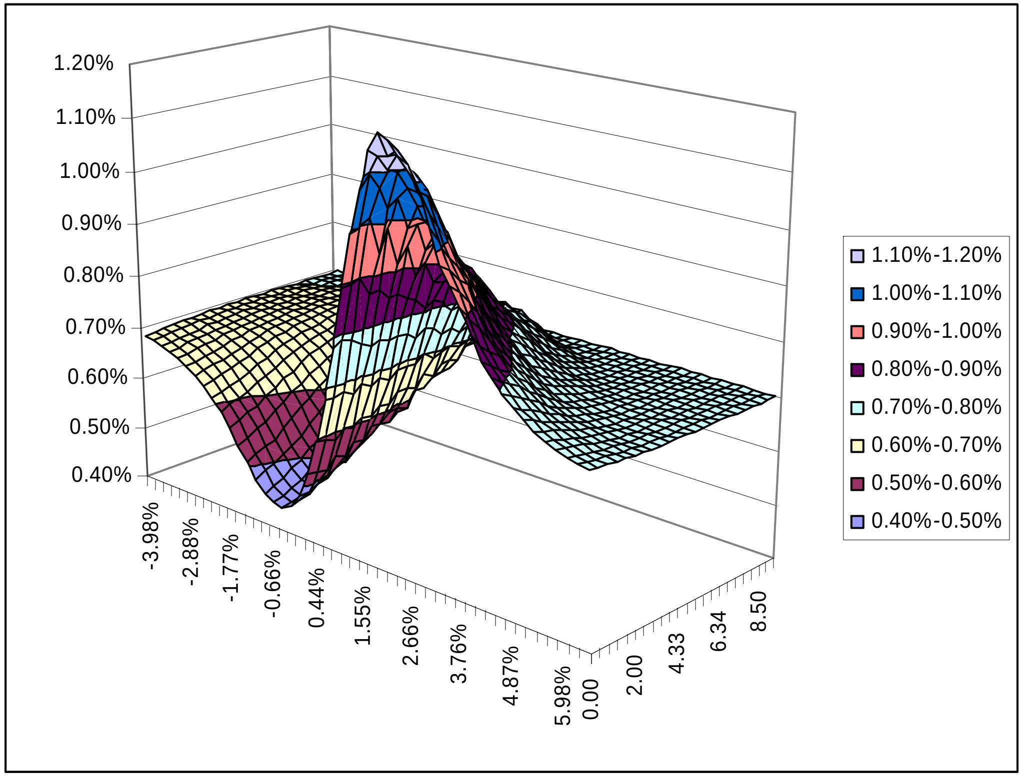
- The hedgers have the right to cancel the swap on a Bermudan basis.

- Parameters: $k = 0.5\%, \Delta k = 1\%, c_1 = 3\%, c_2 = 2\%$ and $\sigma = 0.70\%, \gamma = 10$ multi period investor.

- Price at constant volatility 2.72% -- model price 1.12%. Time to calc = 0.2 seconds.

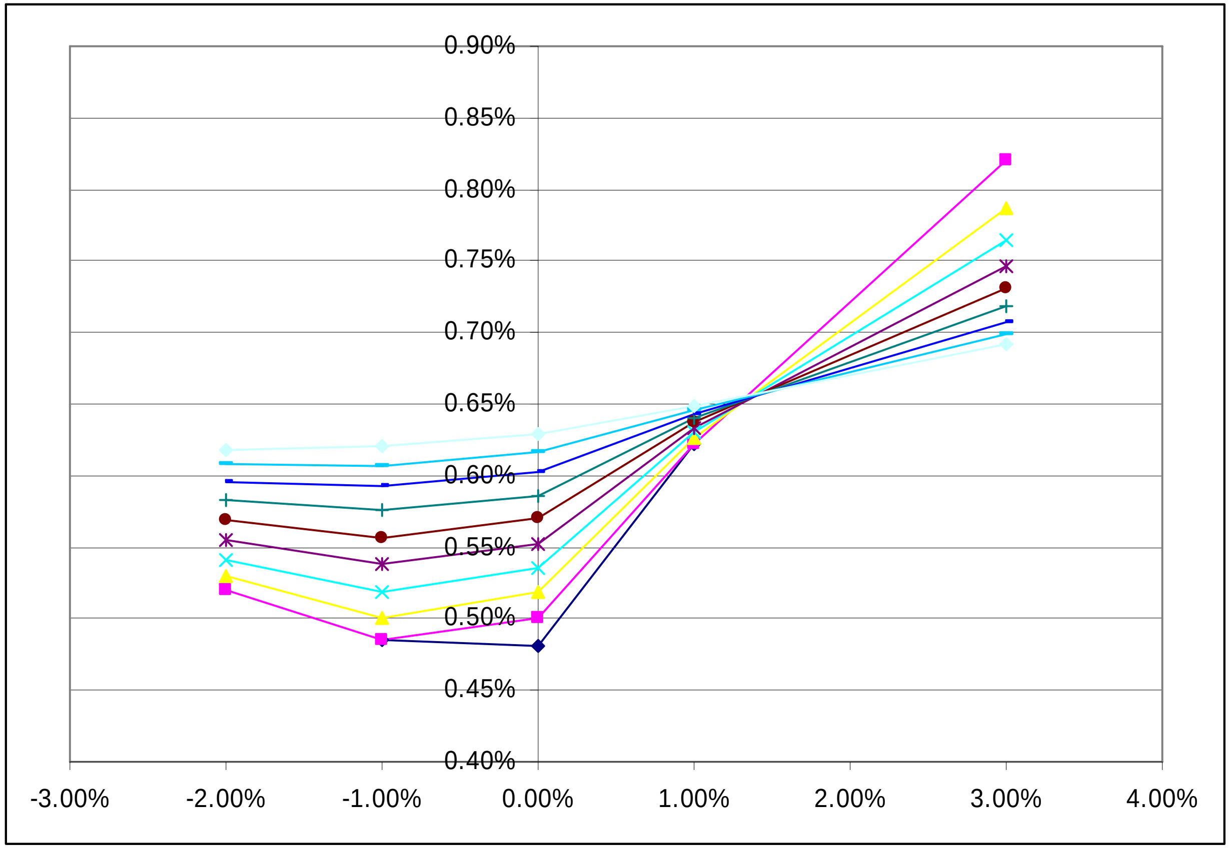
Local Volatility in Example

- Local (normal) volatility as function of time and spot



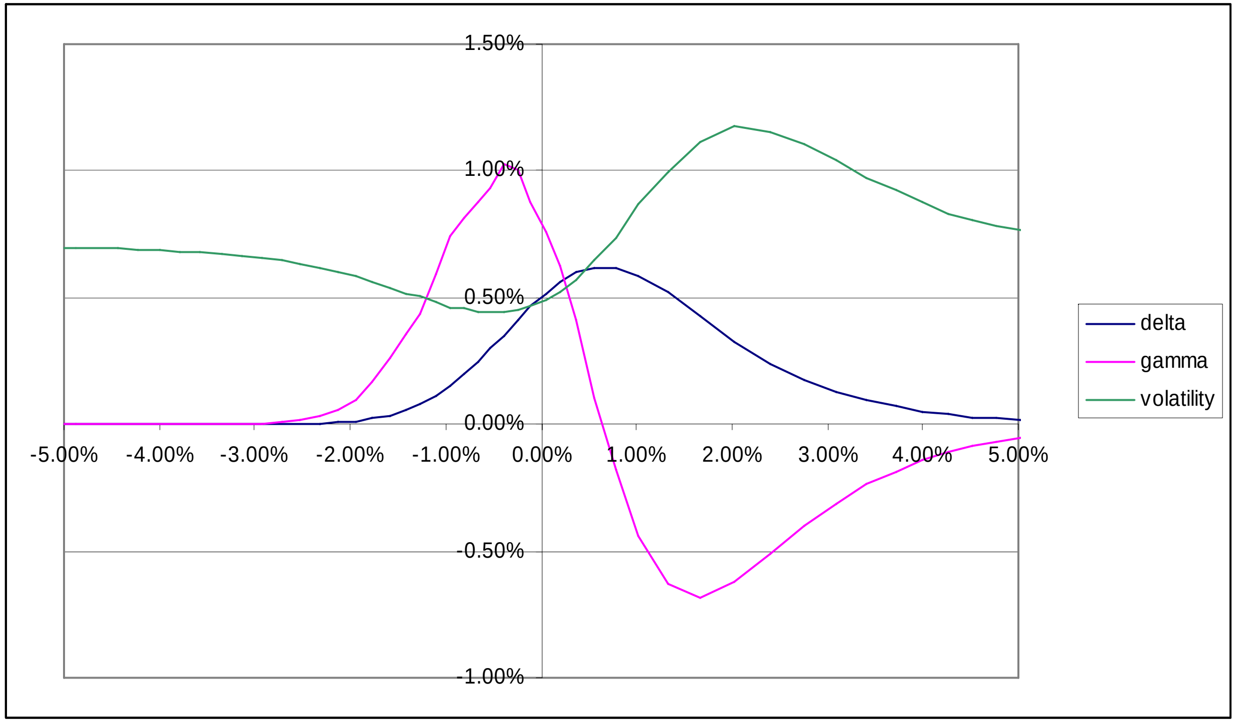
Implied Volatility in Example

- Implied (normal) volatility as function of expiry and strike



Delta, Gamma and Volatility

- Realised volatility and hedgers’ Gamma are inversely related.



Implied Demand

- From Dupire (1994) local volatility from butterfly and term spread on observed option prices:

$$\hat{\vartheta}(t,S) = [2\frac{C_T(T=t,K=S)}{C_{KK}(T=t,K=S)}]^{1/2}$$

- Combining this with previous results yield

$$\frac{b_S(t,S)}{a_x(t,S)} = \frac{\sigma}{\hat{\vartheta}(t,S)} - 1$$

Practical Questions

- Where do we get the hedgers' portfolio from?
  - Use own portfolio as an educated guess.
  - Or we use the implied route.
- How do we estimate risk aversion/market depth/liquidity?
  - Time series data of orders and price (?)
  - Implied volatility smiles.

Conclusion

- If markets have finite liquidity the realised volatility and hedgers' Gamma will be inversely related.
- How much is a function of the risk aversion of fundamental investors and the magnitude of the Gamma of the hedgers.
- As the hedgers are gamma short in some areas and gamma long in other areas this can create volatility skew/smiles.
- Numerical solution is feasible in toy (FD) models but more difficult in more realistic (MC) models – particularly in multi factor models. Without further tricks and developments very difficult to solve model in a Monte-Carlo context.
- Nonetheless, maybe we should devote more time to development of these types of models rather than keep on increasing model complexity with addition of stochastic volatility, jumps, etc.

- Theoretical question: is it possible to solve a “relevant” perpetual problem explicitly?
  [I think it is, but I’ve had no success so far].