Spectral Theory for Stochastic Volatility and Time-Changed Diffusions

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Warwick Mathematical Institute Workshop

July 12, 2008

Introduction and Overview of the talk

- Traditionnally Functional Analysis (Semigroup Theory) and Probability have been considered separately. Reconciling them provides interesting answers, both theoretically and numerically.
- Financially speaking, pricing options (in the Equity world) starts with calibrating a non-flat volatility smile. How to do it?
- First Application: Asymptotics (Long-Maturity) of the implied volatility smile under Heston.
- Second Application: Subordination.

General Notations

- $(X_t)_{t\geq 0}$: Continuous-time Markov Process with values in $\mathbb{R}$ satisfying

$$dX_t = b(X_t)\,dt + a(X_t)\,dW_t$$

  Where W_t is a standard Brownian motion, and $a(.)$ and $b(.)$ such that a unique strong solution exists.
- Φ: Twice continuously differentiable function (payoff).
- $r(.)$: Non-negative function (interest/killing rate).

No Arbitrage Pricing and Semigroups

- Under no-arbitrage conditions, the value at time t of a European claim with payoff Φ at maturity T is worth $(X_t = x)$

$$V(x,t) = \mathbb{E}_{x,t}\left[e^{-\int_t^T r(u)}\Phi(X_T)\right]$$

- Let us define the operator $(P_t\Phi)(x) := V(x,t)$
- Then the operator defines a strongly continuous contraction semigroup:
  - $P_0 = I$
  - $\forall s,t \geq 0,\ \ P_{s+t} = P_s \circ P_t$ (Markov Property)
  - $\forall \Phi \in \Omega,\ \ \lim_{t \to 0} \|P_t\Phi - \Phi\| = 0$
  - $\forall t \geq 0,\ \ \|P_t\| \leq 1$

Infinitesimal Generators

We define the infinitesimal generator $\mathcal{L}$ of the semigroup P_t as

$$\mathcal{L}\Phi(x) = \lim_{t \to 0} \frac{(P_t\Phi)(x) - \Phi(x)}{t}$$

Where

$$\mathcal{D}(\mathcal{L}) = \{\Phi : \mathcal{L}\Phi \text{ exists}\}$$

In our case:

$$\Phi \in \mathcal{D}(\mathcal{L}), \ \mathcal{L}\Phi(x) = b(x)\frac{\partial\Phi}{\partial x} + \frac{1}{2}a^2(x)\frac{\partial^2\Phi}{\partial x^2} - r(x)\Phi(x)$$

From Feynman-Kac to Sturm-Liouville theory

- From the Feynman-Kac formula, the pricing problem is tantamount to solving the Backward Kolmogorov equation

$$\mathcal{L}\Phi = \frac{\partial \Phi}{\partial t}$$

- Take $\Phi(x,t) = e^{\lambda t}\Psi(x)$, the equation reduces to the Sturm-Liouville equation (with appropriate Boundary conditions):

$$\mathcal{L}\Psi = \lambda\Psi$$

- Solution (for the probability density) via the spectral theorem:

$$"\, p(t;x,y) = m(y) \sum_{n\in\mathcal{I}} e^{\lambda_n t}\psi_n(x)\,\psi_n(y)\, "$$

- And hence the option value

$$V(x,t) = \int f(y)\, p(t;x,y)\, dy$$

- Question: Determine $\mathcal{I}, \{\lambda_n, \psi_n()\}_{n\in\mathcal{I}}$

Nature of the spectrum

- Mathematically, the eigenvalues are the poles of the Green function and the eigenfunctions are determined by the residues at these poles.
- The spectrum can be either purely discrete, purely continuous or both.
- The nature of the spectrum is given by the behaviour of the diffusion at the boundary conditions in the Sturm-Liouville equation ($[L, U]$ for Barrier options, $\mathbb{R}_+$ for Call options).
- From a probabilistic point of view, this refers to the Feller classification of end points.
- **Theorem**: When there are no natural or non-oscillating boundaries, then the spectrum is simple and purely discrete.
- We focus here on diffusions with purely discrete spectra: Ornstein-Uhlenbeck, CIR, CEV.
- *Main Drawback: Can't handle Multidimensional diffusions or Jumps.*

How to deal with Stochastic Volatility?

Or the trick to reduce a 2D PDE into a One-Dimensional one.

From a 2D PDE to a 1D PDE (1)

- Consider a general stochastic volatility model under the risk-neutral measure :

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t dW_t \\ dv_t = b(v_t)\, dt + a(v_t)\, dZ_t \\ dW_t dZ_t = \rho dt \end{cases}$$

Where $a(.)$ and $b(.)$ satisfy the usual regularity conditions.

- The 2D PDE for a call option can then be written as :

$$\frac{\partial C}{\partial t} + r\frac{\partial C}{\partial S} + \frac{1}{2} VS^2 \frac{\partial^2 C}{\partial S^2} + b(V)\frac{\partial C}{\partial V} + \frac{1}{2}a^2(V)\frac{\partial^2 C}{\partial V^2} + \rho a(V)\sqrt{V}\frac{\partial^2 C}{\partial S \partial V} - rC = 0$$

With terminal condition $C(S, V, t = T) = (S_T - K)_+$.

- With (a few) changes of variables, one can transform it into

$$\frac{\partial \hat{h}}{\partial \tau} = \frac{1}{2}a^2(V)\frac{\partial^2 \hat{h}}{\partial V^2} + \left[b(V) - ik\rho a(V)\sqrt{V}\right]\frac{\partial \hat{h}}{\partial V} - \frac{k^2 - ik}{2}V\hat{h}$$

With initial condition $\hat{h}(k, V, \tau = 0) = -\frac{K^{ik+1}}{k^2 - ik}$. $\hat{h}$ is - up to a multiplicative term - the Fourier transform of the Call option.

From a 2D PDE to a 1D PDE (2)

- Normalising it, we can show that $\hat{H}$ is related to the call option price by

$$C(S,V,t) = S_t - \frac{Ke^{-r\tau}}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ikx}\frac{\hat{H}(k,V,\tau)}{k^2-ik}dk$$

With $\hat{H}(k,V,\tau=0)=1$

- Using the Separation of variable method and letting $\hat{H}(k,V,\tau) = e^{-\lambda(k)\tau}u(k,V)$, the PDE can be written in the following Eigenvalue form :

$$\mathcal{L}_k u = \lambda(k)\, u$$

Where the Sturm-Liouville operator $\mathcal{L}$ writes

$$\mathcal{L}_k u = -\frac{1}{2}a^2(V)\frac{d^2u}{dV^2} - \left[b(V) - ik\rho a(V)\sqrt{V}\right]\frac{du}{dV} + c(k)\,Vu$$

Analysis for large maturity

- For a Call option written on a stock with dynamics $dS_t = rS_t dt + \sigma S_t dW_t$ (here, σ stands for the implied volatility), we have

$$\frac{C\left(S_t, \sigma, \tau\right)}{Ke^{-r\tau}} \approx_{\tau \to \infty} e^x - \frac{\sqrt{8}}{\sigma\sqrt{\pi\tau}} e^{-\frac{1}{2}d_-^2}$$

- The Fourier form $\hat{H}$ is an analytic function and satisfies the *Ridge Property* (any saddlepoint lies along the imaginary axis). Then, using the Cauchy theorem, we can move the integration contour to the imaginary part of this very saddlepoint. Using a Taylor expansion for $\lambda\left(.\right)$ around the saddlepoint, we obtain

$$\frac{C\left(S, V, \tau\right)}{Ke^{-r\tau}} \approx e^x - \frac{1}{\sqrt{2\pi\lambda''\left(k_0\right)\tau}} \frac{u\left(k_0, V\right)}{k_0^2 - ik_0} e^{-\lambda(k_0)\tau - ik_0 x - \frac{x^2}{2\lambda''(k_0)\tau}}$$

The 'final' formula

- Equating both approximations and expanding in terms of the logmoneyness x, we eventually get

$$V_\tau(x) \approx_{\tau\to\infty} 8\lambda(k_0) - \frac{8}{\tau} ln\left(\frac{u(k_0,V)\sqrt{\lambda(k_0)}}{\left(k_0^2 - ik_0\right)\sqrt{2\lambda''(k_0)}}\right) + (8ik_0+4)\frac{x}{\tau} + \mathcal{O}\left(\frac{1}{\tau^2}\right)$$

- Application to the Heston model $\left(dV_t = \kappa(\theta - V_t)\,dt + \xi\sqrt{V_t}dZ_t\right)$:

$$V_{\tau\to\infty}(0) \approx \frac{4\kappa\theta}{(1-\rho^2)\xi^2}\left\{\sqrt{(2\kappa-\rho\xi)^2 + (1-\rho^2)\xi^2} - (2\kappa-\rho\xi)\right\}$$
$$- \frac{8}{\tau} ln\left[\frac{u(k_0,V)}{\left(k_0^2 - ik_0\right)}\sqrt{\frac{\lambda(k_0)}{2\lambda''(k_0)}}\right] + \mathcal{O}\left(\frac{1}{\tau^2}\right)$$

In particular, the ATM skew is given by

$$\left.\frac{\partial V}{\partial x}\right|_{x=0} \approx_{\tau\to\infty} \frac{1}{\tau}\left\{4 - \frac{8}{1-\rho^2}\left[\frac{1}{2} - \frac{\rho}{\xi}\left(\kappa - \frac{1}{2}\sqrt{4\kappa^2+\xi^2-4\rho\kappa\xi}\right)\right]\right\}$$

Second Approach: Time-Changed Diffusion

- Monroe theorem (1978) : Every semimartingale process can be written as a time-changed Brownian motion.
- Let X_t be a Brownian motion. Consider $Z_t = X_{T_t}$, where T_t is a subordinator, i.e. a non-decreasing positive Levy process, independent of X_t.
- Problem 1 : Determine the properties of the Z-generator.
- Problem 2 : Determine the spectral decomposition of the Z-pdf.

Laplace Exponent for Subordinators

- If T_t is a Lévy process, then its Lévy symbol η is defined as

$$\forall u \in \mathbb{R},\ t \geq 0,\ \mathbb{E}\left[e^{iuT_t}\right] = e^{t\eta(u)}$$

- If T_t is a subordinator (increasing Lévy process starting at 0) then η is a Bernstein function:

$$\eta\left(u\right) = a + ibu + \int_{[0,\infty)} \left(e^{iuy} - 1\right)\lambda\left(dy\right)$$

Where $b \geq 0$ and the Lévy measure λ satisfies

$$\lambda\left(\left(-\infty, 0\right)\right) = 0 \text{ and } \int_{[0,\infty)} \left(1 \wedge y\right)\lambda\left(dy\right) < \infty$$

- Theorem
  Let $f:(0,\infty)\to\mathbb{R}$ be a Bernstein function and $(\eta_t)_{t\geq 0}$ the associated convolution semigroup on $\mathbb{R}$ supported by $\mathbb{R}_+$. Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on the Banach space $(X,\|.\|_X)$ with generator $(\mathcal{A},D(\mathcal{A}))$. Let us define T_t^f by the so-called Bochner integral

$$T_t^f u = \int_0^\infty (T_s u)\,\eta_t(ds)$$

  Then the integral is well defined and $\left(T_t^f\right)_{t\geq 0}$ is a strongly continuous contraction semigroup on X and is called the subordinated semigroup (in the sense of Bochner) of $(T_t)_{t\geq 0}$ with respect to $(\eta_t)_{t\geq 0}$
  - Let $\psi_t(u)=-\eta_t(iu)$ be the Laplace exponent of T_t, then

$$\eta_{Z_t} = -\psi_t\circ(-\eta_X)$$

Example: Gamma Subordinator

Let T_t be a Gamma process with parameters $\alpha, \beta > 0$, with density

$$\forall x \geq 0,\ f_{T_t}(x) = \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} x^{\alpha t - 1} e^{-\beta x}$$

And so

$$\int_0^\infty e^{-ux} f_{T_t}(x)\, dx = \exp\left\{-t \int_0^\infty \left(1 - e^{-ux}\right) \frac{\alpha}{x} e^{-\beta x} dx\right\}$$
$$= \left(1 + \frac{u}{\beta}\right)^{-\alpha t} = \exp\left\{-t\alpha \log\left(1 + \frac{u}{\beta}\right)\right\}$$

Hence T_t is a subordinator with $b = 0$ and $\lambda(dx) = \frac{\alpha}{x} e^{-\beta x}$ and $\psi : u \mapsto \alpha \log\left(1 + \frac{u}{\beta}\right)$ is its Bernstein function.

Spectral Decomposition of Subordinated Processes

Theorem

Let $\mathcal{L}$ be the infinitesimal generator of a symmetric sub-Markovian semigroup $(T_t)_{t\geq 0}$ on $L^2(X, m)$. Let f be a Bernstein function associated to a convolution semigroup $\left(\mu_t^f\right)_{t\geq 0}$. Then the subordinated generator is defined by $\mathcal{L}^f = -f(-\mathcal{L})$, where

$$-f(-\mathcal{L}) = \int_0^\infty f(\lambda)\, d(-P_\lambda)$$

Where $-P_\lambda$ is the projection-valued measure associated to the operator $-\mathcal{L}$. Furthermore $Dom\left(\mathcal{L}^f\right) = Dom(-f(-\mathcal{L}))$.

$dX_t = (a - bX_t)\, dt + \sigma dW_t$

- Eigenvalues: $\forall n \in \mathbb{N},\ \lambda_n = -bn,\ \lambda_n^Z = -f(-\lambda_n)$
- Normalised Eigenfunctions:

$$\forall n \in \mathbb{N},\ \Phi_n(x) = \left(\frac{\sigma\sqrt{b}}{2^{n+1} n! \sqrt{\pi}}\right)^{1/2} H_n\left(\frac{\sqrt{b}}{\sigma}\left(x - \frac{a}{b}\right)\right)$$

- Pdf:

$$p(t; x, y) = m(y) \sum_{n \geq 0} e^{\lambda_n t} \Phi_n(x) \Phi_n(y)$$

- $\forall n \in \mathbb{N},\ \Phi_n^Z(u) = \Phi_n(u)$ and

$$p^Z(t; x, y) = m(y) \sum_{n \geq 0} e^{\lambda_n^Z t} \Phi_n(x) \Phi_n(y)$$

$dX_t = \kappa (\theta - X_t) dt + \sigma \sqrt{X_t} dW_t$

- Eigenvalues: $\forall n \in \mathbb{N},\ \lambda_n = \gamma n + \frac{\beta}{2}(\gamma - \kappa),\ \lambda_n^Z = -f(-\lambda_n)$, where $\gamma = \sqrt{\kappa^2 + 2\sigma^2}$, $\beta = \frac{2\kappa\theta}{\sigma^2}$
- Normalised Eigenfunctions:

$$\forall n \in \mathbb{N},\ \Phi_n(x) = \sqrt{\frac{\sigma^2 n!}{2\Gamma(\beta + n)}} \left(\frac{2\gamma}{\sigma^2}\right)^{\beta/2} e^{\frac{(\kappa-\gamma)x}{\sigma^2}} L_n^{(\beta-1)}\left(\frac{2\gamma}{\sigma^2}x\right)$$

  Where $L_n^{(\beta)}$ are the generalized Laguerre Polynomials.
- Pdf:

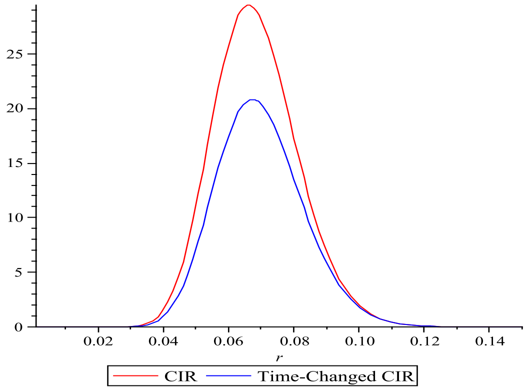
$$p(t; x, y) = m(y) \sum_{n\geq 0} e^{\lambda_n t} \Phi_n(x) \Phi_n(y)$$

- $\forall n \in \mathbb{N},\ \Phi_n^Z(u) = \Phi_n(u)$ and

$$p^Z(t; x, y) = m(y) \sum_{n\geq 0} e^{\lambda_n^Z t} \Phi_n(x) \Phi_n(y)$$

Numerical applications and issues

We take $x_0 = 0.06, \kappa = 2, \theta = 0.07, T = 1, \sigma = 0.1, \alpha = 3, \beta = 0.5$



Conclusion

- Closed form approximation for the implied volatility smile under Heston.
- Semi-closed-form formulae for densities of time-changed one-dimensional diffusions.
- Further research (in progress) :
  - Closed-form approximation fot the Heston IV for a fixed maturity.
  - Other SV models?
  - Straightforward to get a pricing formula and the Greeks.
  - Handle general Lévy processes via subordination of one-dimensional diffusions.

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Speed and scale measures

Consider the diffusion $dX_t = b(X_t)\,dt + a(X_t)\,dW_t$ and define

- Scale Measure:

$$s(x) = \exp\left(-\int^x \frac{2b(y)}{a^2(y)}dy\right)$$

- Speed Measure:

$$m(x) = \frac{2}{a^2(x)\,s(x)}$$

Feller Classification

- Natural : Unattainable from the interior in finite time. The process cant be started there and no boundary conditions needed.
- Regular : The process can enter and leave in finite time. Boundary conditions (Reflection or absorption must be specified).
- Entrance : Unattainable in finite time, but the process can be started there.
- Exit : Reachable from the interior in finite time. Absorbing state.