An n-Dimensional Markov-Functional Interest Rate Model

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Outline

Background

Some models for the pricing and hedging of interest rate derivatives:

- Short rate models: Stable and fast computations; not very flexible w.r.t. calibration, smiles.
- LIBOR market models (LMM): Easy to understand, good intuition of model behaviour, flexible/powerful calibration; Computationally very (too) intense; Benchmark model.
- Markov-functional models (MFM).
  - Introduced 1999 by Hunt, Kennedy & Pelsser.
  - Main intuition: Short rate model efficiency (build on a lattice) combined with the LMM flexibility.
  - Quite popular in the city.
  - Only solved for one- or two-dimensional driving state processes $\rightarrow$ potentially limited correlation structure.

Aim of the paper

- Develop an n-dimensional Markov-functional interest rate model (MFM).
- Investigate similarities and differences between the MFM and the LMM $\rightarrow$ can we transfer the intuition from the LMM SDE to the MFM?
- Investigate potential usefulness in practise: Price Targeted Accrual Redemption Notes (TARNs).
  Currently very popular in the market and are typically priced using multifactor LIBOR market models (LMM).

Notation and Setup

- Set of increasing maturities:
  $\text{today} = 0 < T_1 < T_2 < \cdots < T_n < T_{n+1}$,
- Zero-coupon bonds: D_{tT},
- LIBOR forward rates: L_t^i
- The rolling spot measure, $\mathbb{N}$: The EMM using the discrete savings account as numeraire.

$$N_t = D_{tT_1}, \qquad t \le T_1, \tag{1}$$

$$N_t = D_{tT_{i+1}} \cdot \prod_{j=1}^{i} (1 + \alpha_j L_{T_j}^j), \quad T_i \le t \le T_{i+1}. \tag{2}$$

The LIBOR market model SDE

Let

$$x_t^i = \int_0^t \sigma_s^i dW_s^i, \quad i = 1, \dots, n \tag{3}$$

$$\mathrm{d}W_t^i \mathrm{d}W_t^j = \rho^{ij} \mathrm{d}t \tag{4}$$

Then, under $\mathbb{N}$, each $L_{T_i}^i$ is given by

$$L_{T_i}^i = L_0^i \cdot \exp\left(\int_0^{T_i} \mu(L_t^1, \dots, L_t^i, \sigma, \rho) \mathrm{d}t + x_{T_i}^i \right) \tag{5}$$

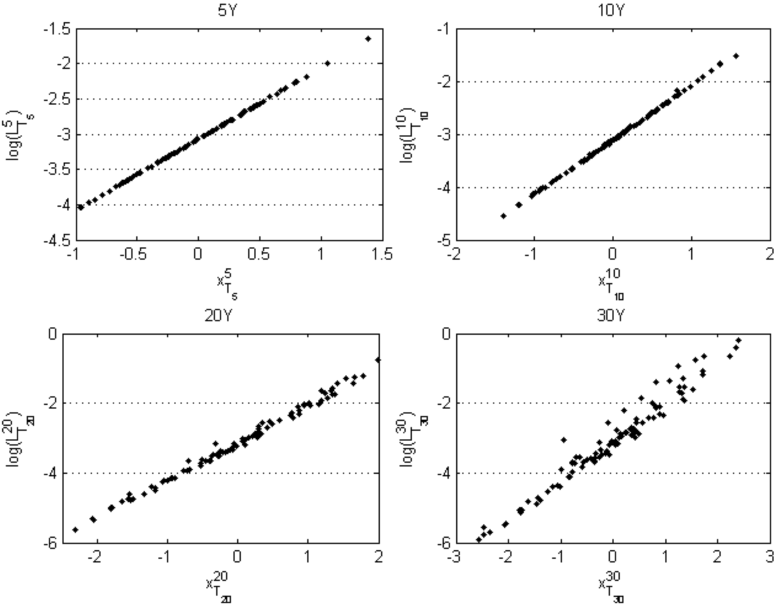
- Note: Stochastic drift term!

The LIBOR market model: Calibration

- σ^i: Under the measure transformation $\mathbb{N} \rightarrow Q^i$, $\mu(L_t^1, \ldots, L_t^i, \sigma, \rho) = 0$. Hence, L_t^i is a lognormal martingale $\rightarrow$ Caplet prices by the Black formula $\rightarrow \sigma^i$ are given directly from market prices of caplets.
- Instantaneous correlations much harder! Due to efficiency one must use approximation formulas:

$$\mathrm{Corr}(\log(L_{T_i}^i), \log(L_{T_j}^j)) \approx \frac{\int_0^{\min(T_i,T_j)} \sigma_t^i \sigma_t^j \rho^{ij} \mathrm{d}t}{\sqrt{\int_0^{T_i} (\sigma_t^i)^2 \mathrm{d}t} \sqrt{\int_0^{T_j} (\sigma_t^j)^2 \mathrm{d}t}}. \quad (6)$$

  - The trader has a view about Terminal Correlations (typically from historical estimation or implied from the Swaptions market) and changes ρ^{ij}s accordingly.
  - Dangerous due to approximation errors?


5Y
10Y
20Y
30Y
-1.5
-2
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-4
-4.5
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1
1.5
$\log(L^5_{T_5})$
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-1
-2
-3
-4
-5
2
$\log(L^{10}_{T_{10}})$
$x^{10}_{T_{10}}$
-6
$\log(L^{20}_{T_{20}})$
$x^{20}_{T_{20}}$
3
$\log(L^{30}_{T_{30}})$
$x^{30}_{T_{30}}$

Postulate

$$L^i_{T_i} = f^i(x^i_{T_i}), \quad i = 1, \dots, n \tag{7}$$

where f^i is some **monotone** function. The functional forms will be found (numerically) by forward induction, forcing the model to be

- arbitrage free, and
- calibrated to Black's formula for Caplets.

To find the functional forms we use digital Caplets in Arrears (DCiA). Value of a DCiA at time 0 under $\mathbb{N}$:

$$V_i(K) = N_0 E^{\mathbb{N}} \left[\frac{\mathbf{1}\{L^i_{T_i} \geq K\}}{N_{T_i}} \right]. \tag{8}$$

- Fact: Pricing DCiA (of all strikes) are equivalent to pricing digital Caplets and Caplets.

Construction: Step 1/3

- Suppose we would like to know $f^i(x^*)$.

Define

$$J^i(x^*) = N_0 E^{\mathbb{N}} \left[\frac{\mathbf{1}\{x^i_{T_i} \geq x^*\}}{N_{T_i}} \right]. \tag{9}$$

Compute the expectation by Monte Carlo integration

$$J^i(x^*) \approx N_0 \frac{1}{m} \sum_{k=1}^{m} \frac{\mathbf{1}\{x^i_{T_i}(\omega_k) \geq x^*\}}{\prod_{l=1}^{i-1}(1 + \alpha_l f^l(x^l_{T_l}(\omega_k)))}. \tag{10}$$

Construction: Step 2/3

- Market prices of DCiA must be consistent with an arbitrage free model.
- Want to mimick the lognormal LMM $\rightarrow$ choose the Black model.

Search for the strike $K(x^*)$ such that

$$V^i(K(x^*)) = J^i(x^*) \tag{11}$$

Construction: Step 3/3

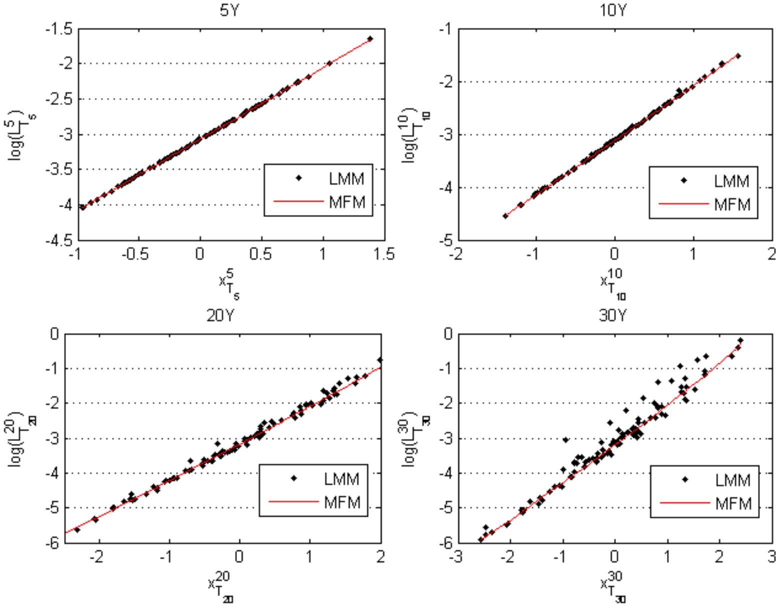
Conclude that

$$f^i(x^*) = K(x^*) \tag{12}$$

WHY:

$$\begin{aligned}
N_0 E^{\mathbb{N}}\left[\frac{\mathbf{1}\{L^i_{T_i} \geq K(x^*)\}}{N_{T_i}}\right] &= V^i(K(x^*)) = \\
J^i(x^*) &= N_0 E^{\mathbb{N}}\left[\frac{\mathbf{1}\{x^i_{T_i} \geq x^*\}}{N_{T_i}}\right] = \\
N_0 E^{\mathbb{N}}\left[\frac{\mathbf{1}\{f^i(x^i_{T_i}) \geq f^i(x^*)\}}{N_{T_i}}\right] &= N_0 E^{\mathbb{N}}\left[\frac{\mathbf{1}\{L^i_{T_i} \geq f^i(x^*)\}}{N_{T_i}}\right],
\end{aligned}$$

- Note: The monotonicity assumption is crucial.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$\log(L^{30}_{T_{30}})$
$x^{30}_{T_{30}}$

TARNs: definition

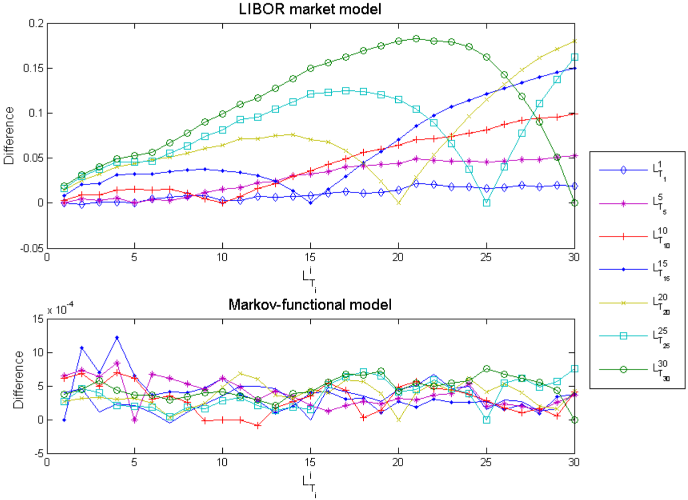
- Want to price TARN swaps.
- Receive: Structured coupon; $\max(10\% - 2L^i_{T_i}, 0)$.
- Pay: $2L^i_{T_i}$.
- Continue until final maturity OR when total received coupon is 10 %.
- Need models with good views on Terminal correlations.

TARN prices

- initial LIBORs $L_0^i = \max(2\% + 0.5\% T_i, 10\%)$
- $\sigma_t^i = 20\%, \forall t, i$
- Instantaneous correlations $\rho^{ij} = \exp\{-0.05|T_i - T_j|\}$
- Notional 10 000.

	5	10	15	20	25	30
LMM	-187.6	-729.2	-1095.3	-1270.5	-1338.0	-1362.5
MFM	-186.5	-719.2	-1067.4	-1230.9	-1294.0	-1316.7
vega	10.3	21.6	20.6	14.9	10.3	7.5
corr	-1.7	-9.6	-18.2	-22.4	-24.4	-25.2

Terminal correlations



Matching the models

Idea: Matching the models Terminal correlations → similar properties.
Test: Change the ρ^{ij}s for the MFM s.t. it matches the simulated Terminal correlations of the LMM.
Results:

	5	10	15	20	25	30
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MFM	-186.5	-719.2	-1067.4	-1230.9	-1294.0	-1316.7
$\widehat{\mathrm{MFM}}$	-187.0	-726.5	-1090.3	-1266.6	-1337.9	-1365.9
vega	10.3	21.6	20.6	14.9	10.3	7.5
corr	-1.7	-9.6	-18.2	-22.4	-24.4	-25.2

TARN pricing summary

- Calibrate both models using the approximation formula $\longrightarrow$
  - The MFM will give prices consistent with the formula.
  - The LMM will not.
- Need a better approximation formula in order to calibrate the LMM satisfactory.
- For the MFM this is straightforward.

Punchlines

With the n-dimensional Markov-functional model we have a model that is

- Very similar in spirit to the n-factor LIBOR market model.
- Arbitrage free
- Calibrated to the Caplet market

Moreover it resolves/improves two major problems with n-factor LIBOR market models

- Calibration to Terminal correlations.
- Computation times (The MFM is up to 40 times faster in my implementation).

THANK YOU!