An $n$-Dimensional Markov-Functional Interest Rate Model

Linus Kaisajuntti$^1$ Joanne Kennedy$^2$

$^1$Department of Finance, Stockholm School of Economics
$^2$Department of Statistics, University of Warwick

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Outline

Introduction

The $n$-dimensional Markov-functional model

Pricing tests

TARNs and Terminal correlations
Background

Some models for the pricing and hedging of interest rate derivatives:

► Short rate models: Stable and fast computations; not very flexible w.r.t. calibration, smiles.

► LIBOR market models (LMM): Easy to understand, good intuition of model behaviour, flexible/powerful calibration; Computationally very (too) intense; Benchmark model.

► Markov-functional models (MFM).
  ► Introduced 1999 by Hunt, Kennedy & Pelsser.
  ► Main intuition: Short rate model efficiency (build on a lattice) combined with the LMM flexibility.
  ► Quite popular in the city.
  ► Only solved for one- or two-dimensional driving state processes → potentially limited correlation structure.
Aim of the paper

- Develop an $n$-dimensional Markov-functional interest rate model (MFM).
- Investigate similarities and differences between the MFM and the LMM → can we transfer the intuition from the LMM SDE to the MFM?
- Investigate potential usefulness in practise: Price Targeted Accrual Redemption Notes (TARNs). Currently very popular in the market and are typically priced using multifactor LIBOR market models (LMM).
Notation and Setup

- Set of increasing maturities:
  \( \text{today} = 0 < T_1 < T_2 < \cdots < T_n < T_{n+1}, \)
- Zero-coupon bonds: \( D_{tT} \)
- LIBOR forward rates: \( L_t^i \)
- The rolling spot measure, \( N \): The EMM using the discrete savings account as numeraire.

\[
N_t = D_{tT_1}, \quad t \leq T_1, \tag{1}
\]
\[
N_t = D_{tT_{i+1}} \cdot \prod_{j=1}^{i} (1 + \alpha_j L_{T_j}^j), \quad T_i \leq t \leq T_{i+1}. \tag{2}
\]
The LIBOR market model SDE

Let

$$x_t^i = \int_0^t \sigma_s^i dW_s^i, \quad i = 1, \ldots, n$$

(3)

$$dW_t^i dW_t^j = \rho^{ij} dt$$

(4)

Then, under \( \mathbb{N} \), each \( L_{T_i}^i \) is given by

$$L_{T_i}^i = L_0^i \cdot \exp \left( \int_0^{T_i} \mu(L_t^1, \ldots, L_t^i, \sigma, \rho) dt + x_{T_i}^i \right)$$

(5)

▶ Note: Stochastic drift term!
The LIBOR market model: Calibration

- $\sigma^i$: Under the measure transformation $\mathbb{N} \rightarrow Q^i$, 
  $\mu(L^1_t, \ldots, L^i_t, \sigma, \rho) = 0$. Hence, $L^i_t$ is a lognormal martingale 
  → Caplet prices by the Black formula → $\sigma^i$ are given directly 
  from market prices of caplets.

- Instantaneous correlations much harder! Due to efficiency one 
  must use approximation formulas:

$$\text{Corr}(\log(L^i_{T_i}), \log(L^j_{T_j})) \approx \frac{\int_0^{\min(T_i, T_j)} \sigma_t \sigma_t \rho_{ij} \, dt}{\sqrt{\int_0^{T_i} (\sigma_t^i)^2 \, dt} \sqrt{\int_0^{T_j} (\sigma_t^j)^2 \, dt}}.$$  \hspace{1cm} (6)

- The trader has a view about Terminal Correlations (typically 
  from historical estimation or implied from the Swaptions 
  market) and changes $\rho_{ij}$'s accordingly.
- Dangerous due to approximation errors?
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Postulate

\[ L^i_{T_i} = f^i(x^i_{T_i}), \quad i = 1, \ldots, n \]  

(7)

where \( f^i \) is some **monotone** function. The functional forms will be found (numerically) by forward induction, forcing the model to be

- arbitrage free, and
- calibrated to Black’s formula for Caplets.

To find the functional forms we use digital Caplets in Arrears (DCiA). Value of a DCiA at time 0 under \( \mathbb{N} \):

\[ V_i(K) = N_0 E^{\mathbb{N}} \left[ \frac{1\{L^i_{T_i} \geq K\}}{N_{T_i}} \right] . \]  

(8)

- Fact: Pricing DCiA (of all strikes) are equivalent to pricing digital Caplets and Caplets.
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Construction: Step 1/3

- Suppose we would like to know \( f^i(x^*) \).

Define

\[
J^i(x^*) = N_0 E^\mathbb{N} \left[ \frac{\mathbf{1}\{x^i_{T_i} \geq x^*\}}{N_{T_i}} \right].
\]  

(9)

Compute the expectation by Monte Carlo integration

\[
J^i(x^*) \approx N_0 \frac{1}{m} \sum_{k=1}^{m} \frac{\mathbf{1}\{x^i_{T_i}(\omega_k) \geq x^*\}}{\prod_{l=1}^{i-1}(1 + \alpha_l f^l(x^l_{T_l}(\omega_k)))}.
\]  

(10)
Construction: Step 2/3

Market prices of DCiA must be consistent with an arbitrage free model.

Want to mimick the lognormal LMM → choose the Black model.

Search for the strike $K(x^*)$ such that

$$V^i(K(x^*)) = J^i(x^*)$$  \hspace{1cm} (11)
Construction: Step 3/3

Conclude that

\[ f^i(x^*) = K(x^*) \]  \hspace{1cm} (12)

WHY:

\[ N_0 E^N \left[ \frac{1\{L^i_{T_i} \geq K(x^*)\}}{N_{T_i}} \right] = V^i(K(x^*)) = \]

\[ J^i(x^*) = N_0 E^N \left[ \frac{1\{x^i_{T_i} \geq x^*\}}{N_{T_i}} \right] = \]

\[ N_0 E^N \left[ \frac{1\{f^i(x^i_{T_i}) \geq f^i(x^*)\}}{N_{T_i}} \right] = N_0 E^N \left[ \frac{1\{L^i_{T_i} \geq f^i(x^*)\}}{N_{T_i}} \right], \]

▶ Note: The monotonicity assumption is crucial.
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TARNs: definition

- Want to price TARN swaps.
- Receive: Structured coupon; \( \max(10\% - 2L^iT_i, 0) \).
- Pay: \( 2L^iT_i \).
- Continue until final maturity OR when total received coupon is 10%.
- Need models with good views on Terminal correlations.
TARN prices

- initial LIBORs $L^i_0 = \max(2\% + 0.5\% T_i, 10\%)$
- $\sigma^i_t = 20\%, \forall t, i$
- Instantaneous correlations $\rho^{ij} = \exp\{-0.05|T_i - T_j|\}$
- Notional 10 000.

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Terminal correlations

**LIBOR market model**

**Markov-functional model**

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Matching the models

Idea: Matching the models Terminal correlations $\rightarrow$ similar properties.
Test: Change the $\rho_{ij}$s for the MFM s.t. it matches the simulated Terminal correlations of the LMM.
Results:

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TARN pricing summary

- Calibrate both models using the approximation formula →
  - The MFM will give prices consistent with the formula.
  - The LMM will not.
- Need a better approximation formula in order to calibrate the LMM satisfactory.
- For the MFM this is straightforward.
Punchlines

With the $n$-dimensional Markov-functional model we have a model that is

- Very similar in spirit to the $n$-factor LIBOR market model.
- Arbitrage free
- Calibrated to the Caplet market

Moreover it resolves/improves two major problems with $n$-factor LIBOR market models

- Calibration to Terminal correlations.
- Computation times (The MFM is up to 40 times faster in my implementation).
THANK YOU!